Deck Transformations

Analogy: Galois theory.

given a finite field ext $i: F \hookrightarrow E$

$$\mathrm{Aut}(E/F) = \left\{ \varphi : E \to E \;\middle|\; \varphi|_F = \mathrm{id} \quad (\text{or } \varphi \circ i = i) \right\}$$

e.g. $\mathbb{Q} \hookrightarrow \mathbb{Q}[\sqrt{2}, \sqrt{3}]$

$\mathrm{Aut} = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\quad \sqrt{2} \longmapsto \pm\sqrt{2}$

$\quad \sqrt{3} \longmapsto \pm\sqrt{3}$

$\mathbb{Q} \hookrightarrow \mathbb{Q}[\sqrt[3]{2}]$

$\mathrm{Aut} = 1$

intermediate subfields $\qquad\qquad$ subgroups

$$\begin{array}{ccc} E & & 1 \\ \uparrow & & \cap \\ K & & H \\ \uparrow & & \cap \\ F & & G = \mathrm{Aut}(E/F) \end{array}$$

$$K \longmapsto \mathrm{Aut}(E/K)$$

$$\text{fixed field} \longleftarrow\!\!\!\shortmid \; H$$

$\{ e \in E \mid \varphi(e) = e \;\; \forall \varphi \in H \}$

E/F is Galois if
$|\mathrm{Aut}(E/F)| = [E:F]$ (not smaller)
or equivalently
fixed field of $\mathrm{Aut}(E/F)$ is F (not bigger)

then corresp. above is a bijection

E/K is Galois $\forall$ intermediate field K

if K corresp. to $H \trianglelefteq G$ then K/F is Galois
~~And~~ $\mathrm{Aut}(K/F) = G/H$.

Back to covering spaces

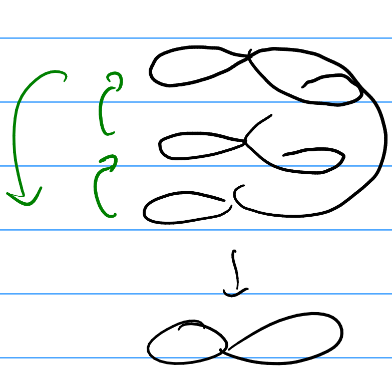
given $p: \tilde{X} \to X$, define

$$\mathrm{Aut}(\tilde{X}/X) = \{ \varphi : \tilde{X} \to \tilde{X} \mid p \circ \varphi = p \}$$

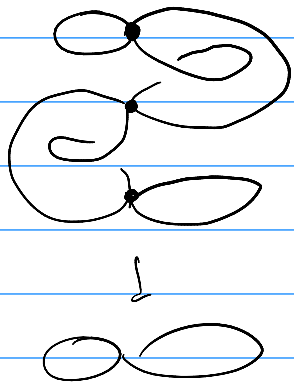
$$\begin{tikzcd} \tilde{X} \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

"group of deck transformations"

examples:



$\mathrm{Aut} = \mathbb{Z}/3$



$\mathrm{Aut} = 1$

Correspondence

intermediate covers $\qquad\qquad\qquad$ subgroups

$$\begin{tikzcd} \tilde{X} \arrow[dr, "\tilde{p}"] \arrow[dd, "p"'] & \\ & X' \arrow[dl, "p'"] \\ X & \end{tikzcd} \qquad\qquad \begin{matrix} 1 \\ \cap \\ H \\ \cap \\ G = \mathrm{Aut}(\tilde{X}/X) \end{matrix}$$

$$X' \longmapsto \mathrm{Aut}(\tilde{X}/X') \subset G$$

$$\tilde{X}/H \longleftarrow\!\!\!| \ H$$

analogue of a Galois extension
is a <u>normal cover</u>:

either $\mathrm{Aut}(\tilde{X}/X) \longrightarrow$ fiber $p^{-1}(x_0)$
is a bij. and not just inj.

or $\tilde{X}/\mathrm{Aut} = X$ and not bigger

or $p_*\pi_1(\tilde{X}) \triangleleft \pi_1(X)$

then · $\mathrm{Aut} = \pi_1(X)/p_*\pi_1(\tilde{X})$
· corresp is a bijection.
· $\tilde{X}/X'$ is normal $\forall$ intermediate X'

· if $\underset{H}{\underset{\shortparallel}{\mathrm{Aut}(X'/X)}} \triangleleft \underset{G}{\underset{\shortparallel}{\mathrm{Aut}(\tilde{X}/X)}}$ then $X' \to X$ is normal
And $\mathrm{Aut}(X'/X) = G/H$

Thm: let $p : \tilde{X} \to X$ be a cover. choose base points
then TFAE:

1. $p_* \pi_1(\tilde{X}) \triangleleft \pi_1(X)$

2. the map $\operatorname{Aut}(\tilde{X}/X) \longrightarrow p^{-1}(x_0)$, $\varphi : \tilde{X} \circlearrowleft \longmapsto \varphi(\tilde{x}_0)$ is a bijection

   always injective.

3. $\tilde{X}/\operatorname{Aut} = X$ not bigger

Pf ① ⇒ ②

let $\varphi, \varphi' \in \operatorname{Aut}(\tilde{X})$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ \tilde{X} \arrow[ur, shift left, "{\varphi, \varphi'}"] \arrow[ur, shift right] \arrow[r, "p"'] & X \end{tikzcd}$$

if $\varphi(\tilde{x}_0) = \varphi'(\tilde{x}_0)$
then $\varphi = \varphi'$ by last time

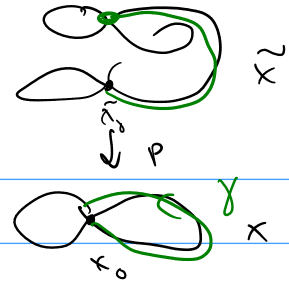
given $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$, we want $\varphi : \tilde{X} \to \tilde{X}$ such that $\varphi(\tilde{x}_0) = \tilde{x}_1$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ \tilde{X} \arrow[ur, dashed, "\varphi"] \arrow[r, "p"'] & X \end{tikzcd}$$

know that φ exists iff

$$p_* \pi_1(\tilde{X}, \tilde{x}_0) \subseteq p_* \pi_1(\tilde{X}, \tilde{x}_1)$$

also know these two are conjugate subgroups
so if they're normal, they're the same.



② ⟹ ① we've seen that

$$\begin{aligned} \pi_1(X,x_0) &\longrightarrow p^{-1}(x_0) \\ [\gamma] &\longmapsto \tilde{\gamma}(1) \\ p_*\pi_1(\tilde{X},\tilde{x}_0) &\rightsquigarrow \tilde{x}_0 \quad (\text{stabilizer}) \end{aligned}$$

is surjective

$$\begin{tikzcd} & \mathrm{Aut}(\tilde{X}/X) \arrow[d, "\text{assumed bijective}"] \\ \pi_1(X,x_0) \arrow[ur, dashed, "\text{get a group hom here}"] \arrow[r, two heads] & p^{-1}(x_0) \end{tikzcd}$$

ker of hom. is $p_*\pi_1(\tilde{X},\tilde{x}_0)$
so that must have been a normal subgroup.

Bonus: $\mathrm{Aut}(\tilde{X}/X) = \pi_1(\tilde{X}) / p_*\pi_1(X)$

② ⟺ ③

$$\begin{tikzcd} \tilde{X} \arrow[d, "p"'] \arrow[r, two heads] & \tilde{X}/\mathrm{Aut} \arrow[dl] \\ X & \end{tikzcd}$$

fiber of this over x_0 is

$$p^{-1}(x_0) / \mathrm{Aut}$$

this = one point iff
Aut acts transitively on $p^{-1}(x_0)$

sketchy but ☒

Thm if $p: \tilde{X} \to X$ is a normal cover then

① corresp. between intermediate covers and $H \leq G = \mathrm{Aut}(\tilde{X}/X)$ is a bijection

② $\forall$ $\tilde{X} \xrightarrow{\tilde{p}} X' \xrightarrow{p'} X$, have

(a) $\tilde{X} \to X'$ is normal

(b) if $\mathrm{Aut}(\tilde{X}/X') \trianglelefteq \mathrm{Aut}(\tilde{X}/X)$ then $X' \to X$ is normal, and $\mathrm{Aut}(X'/X) = G/H$.

Pf ② is easier: consider

$$p_* \pi_1(\tilde{X}) \leq p'_* \pi_1(X') \leq \pi_1(X)$$

ⓐ if $p_* \pi_1(\tilde{X})$ is normal in $\pi_1(X)$ then normal in $p'_* \pi_1(X')$

ⓑ use lattice iso thm: divide by $p_* \pi_1(\tilde{X})$, get

$$1 \leq \frac{p'_* \pi_1(X')}{p_* \pi_1(\tilde{X})} \leq \frac{\pi_1(X)}{p_* \pi_1(\tilde{X})}$$

aka $1 \leq \mathrm{Aut}(\tilde{X}/X') \overset{``H}{} \leq \mathrm{Aut}(\tilde{X}/X) \overset{``G}{}$

and then $\mathrm{Aut}(X'/X) = \dfrac{\pi_1(X)}{\pi_1(X')} = \dfrac{\pi_1(X) / \pi_1(\tilde{X})}{\pi_1(X') / \pi_1(\tilde{X})} = \dfrac{G}{H}$

$$\begin{tikzcd} \tilde X \arrow[r, "\tilde p"] \arrow[d, "p"'] & X' = X/H \arrow[dl, "p'"] \\ X & \end{tikzcd}$$

(1) given $H \leq G$, let $X' = \tilde X / H$

want to show $\mathrm{Aut}(\tilde X / X') = H$ and not bigger

H is naturally a subgroup

$$H \leq \mathrm{Aut}(\tilde X / X') \hookrightarrow \tilde p^{-1}(\text{point}) \quad \}$$

but by def of quot map $\tilde X \to \tilde X / H$,

the map $H \to \tilde p^{-1}(\text{point})$ is surjective.

so they're all the same

Second, given $\begin{tikzcd} \tilde X \arrow[r, "\tilde p"] \arrow[rr, bend right, "p"'] & X' \arrow[r, "p'"] & X \end{tikzcd}$ intermediate cover,

let $H = \mathrm{Aut}(\tilde X / X')$

want $\tilde X / H = X'$ not bigger

or equivalently $\tilde p_* \pi_1(\tilde X) \trianglelefteq \pi_1(X')$

look at $p'_* \tilde p_* \pi_1(\tilde X) \leq p'_* \pi_1(X') \leq \pi_1(X)$

this is normal

so this is normal. ∎