Last time: stated lifting criterion

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & (X, x_0) \end{tikzcd}$$

covering space

$\tilde{f}$ exists iff $f_* \pi_1(Y, y_0) \subset p_* \pi_1(\tilde{X}, \tilde{x}_0)$

$\Rightarrow$ is easy.

$\Leftarrow$ we'll prove today

Given $y \in Y$ choose a path γ from y_0 to y

let $\tilde{\gamma}$ be the lift of $f \circ \gamma$ starting from x_0

$$\begin{tikzcd} & & \tilde{X} \arrow[d, "p"] \\ I \arrow[r, "\gamma"'] \arrow[urr, "\tilde{\gamma}"] & Y \arrow[r, "f"'] & X \end{tikzcd}$$

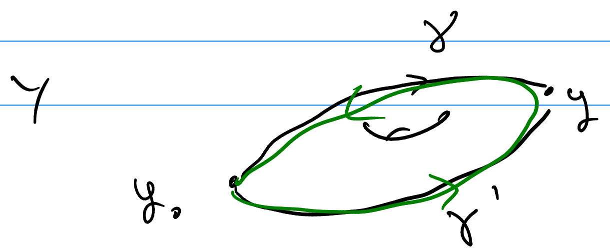
set $\tilde{f}(y) = \tilde{\gamma}(1)$

then $p \circ \tilde{f} = f$ because $p(\tilde{f}(y)) = p(\tilde{\gamma}(1))$

$= \gamma(1) = y$

$\forall y \in Y$

Is $\tilde{f}$ well-defined?



why should $\tilde{\gamma}(1) = \tilde{\gamma}'(1)$

where $\tilde{\gamma}'$ lifts $f \circ \gamma$ ... ?

Let γ' be another path from y_0 to y
let $\tilde{\gamma}'$ be the lift of $f \circ \gamma$ starting from x_0

then $\gamma' \cdot \gamma^{-1}$ is a loop in Y, based at y_0

so $f_*[\gamma' \cdot \gamma^{-1}] = p_*[\beta]$

for some loop β in $\tilde{X}$ based at $\tilde{x}_0$
(using hypothesis finally)

so $f \circ (\gamma' \cdot \gamma^{-1}) = (f \circ \gamma') \cdot (f \circ \gamma)^{-1} \simeq p \circ \beta$

so $f \circ \gamma' \simeq (p \circ \beta) \cdot (f \circ \gamma)$

so their lifts $\tilde{\gamma}' \simeq \beta \cdot \tilde{\gamma}$ are htpic rel. endpoints

so they have the same endpoint:
$\tilde{\gamma}'(1) = \tilde{\gamma}(1)$ ✓

$\tilde{f}$ continuous: on worksheet.

Uniqueness: suppose $f : Y \longrightarrow X$
if two lifts $\tilde{f}$ and $\tilde{f}'$
agree at $y_0 \in Y$
then they're equal.

Pf: Know for $Y = I$ (unique path lifting)

In general, let $y \in Y$
Y is path-conn $\leadsto$ choose path γ from y_0 to y

Then $\tilde{f} \circ \gamma$ and $\tilde{f}' \circ \gamma$ are two paths in $\tilde{X}$

that lift $f \circ \gamma$ and agree at $t = 0$
(start from $\tilde{f}(y_0) = \tilde{f}'(y_0)$)

so agree $\forall t$. at $t = 1$, get $\tilde{f}(y) = \tilde{f}'(y)$

$y \in Y$ was arbitrary, so $\tilde{f} = \tilde{f}'$. $\square$

Can we get Borsuk-Ulam thm?

$$
\begin{tikzcd}
S^2 \arrow[r, "g"] \arrow[d, "q"', "2:1"] \arrow[dr, "p \circ g"] & S^1 \ni z \arrow[d, "p"] \\
\mathbb{RP}^2 \arrow[r, dashed, "\exists h"'] & S^1 \ni z^2
\end{tikzcd}
$$

odd $g(-x) = -g(x)$

$p(g(-x)) = p(g(x))$

$\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2 \xrightarrow{0} \pi_1(S^1) = \mathbb{Z}$

so h lifts to any cover of S^1

h is nullhomotopic... contradiction somewhere?

think about it...