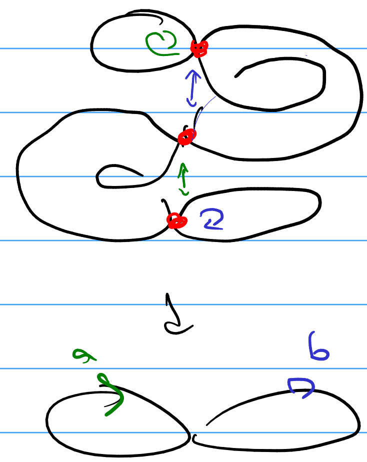
$p: \tilde{X} \to X$ a covering space.

everything is path conn. + loc. path conn.

Unique path lifting, homotopy lifting.

Saw: $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.

$\pi_1(X, x_0)$ acts on the fiber $p^{-1}(x_0)$



Let $[\gamma] \in \pi_1(X, x_0)$ some $\gamma : I \to X$

$\gamma(0) = \gamma(1) = x_0$

let $\tilde{x}_0 \in p^{-1}(x_0)$

define $[\gamma] \cdot \tilde{x}_0 = \tilde{\gamma}(1)$ where

$\tilde{\gamma} : I \to \tilde{X}$

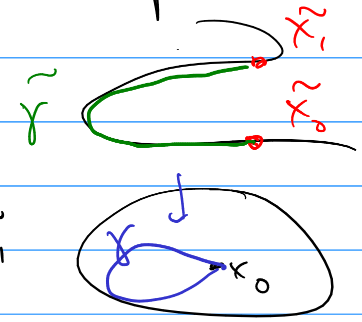
is the unique path with

$p \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(0) = \tilde{x}_0$

check: well-defined, group action

also: action is transitive bec. $\tilde{X}$ is path-conn.

given $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$



choose a path $\tilde{\gamma}$ from $\tilde{x}_0$ to $\tilde{x}_1$

then $\gamma := p \circ \tilde{\gamma}$ is a loop based at x_0

then $[\gamma] \cdot \tilde{x}_0 = \tilde{x}_1$

last: stab. of $\tilde{x}_0 \in p^{-1}(x_0) \overset{\text{def}}{=} \{ [\gamma] \in \pi_1(X, x_0) :$ $\tilde{\gamma}$ with $\tilde{\gamma}(0) = \tilde{x}_0$ also has $\tilde{\gamma}(1) = \tilde{x}_0 \}$

is $p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right) \subset \pi_1(X, x_0)$

Corollary: index of $p_* \pi_1(\tilde{X}) \subset \pi_1(X)$

= # sheets of the cover.

(follows from orbit-stabilizer theorem)

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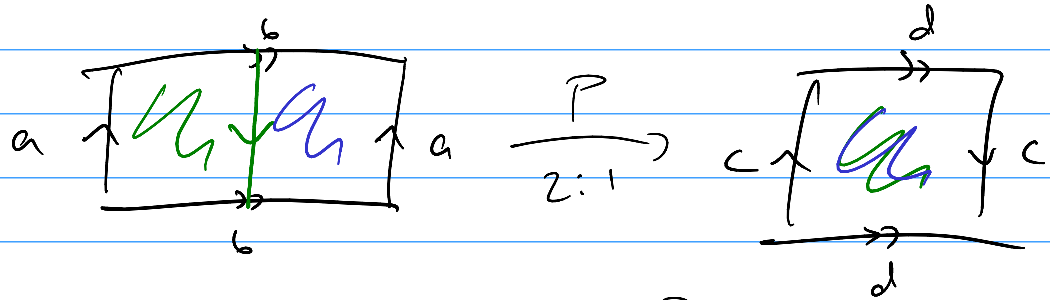
torus $\longrightarrow$ klein bottle



$\pi_1 = \langle a, b \mid ab = ba \rangle \xrightarrow{p_*} \pi_1 = \langle c, d \mid \underbrace{cdcd^{-1} = 1}_{} \rangle$

$a \longmapsto c$

$b \longmapsto d^2$

$ab \longmapsto cd^2$

$ba \longmapsto d^2 c$) these are equal!

injective. image is index 2.

not obvious!

$\left[\text{ker of } \pi_1(K) \longrightarrow \mathbb{Z}/2 \right.$

$c \longmapsto 0$

$\left. d \longmapsto 1 \quad ? \right]$

Another proof that $\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2$ for $n \geq 2$

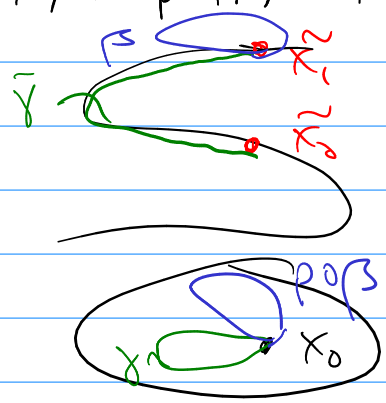
cover $S^n \xrightarrow{2:1} \mathbb{RP}^n$ $\qquad \pi_1(S^n) = 1$ Gee. $n \geq 2$

$p_* \pi_1(S^n) \subset \pi_1(\mathbb{RP}^n)$ index 2

so $|\pi_1(\mathbb{RP}^n)| = 2$

What if we change basepoints?

$p: \tilde{X} \to X \qquad x_0 \in X \qquad \tilde{x}_0, \hat{x}_1 \in p^{-1}(x) \subset \tilde{X}$



compare $p_* \pi_1(\tilde{X}, \tilde{x}_0) \subset \pi_1(X, x_0)$

to $p_* \pi_1(\tilde{X}, \tilde{x}_1) \subset$ same

choose a path $\tilde{\gamma}$ from $\tilde{x}_0$ to $\tilde{x}_1$

let $\gamma = p \circ \tilde{\gamma}$, a loop in X based at x_0

if $[\beta] \in \pi_1(\tilde{X}, \tilde{x}_1)$

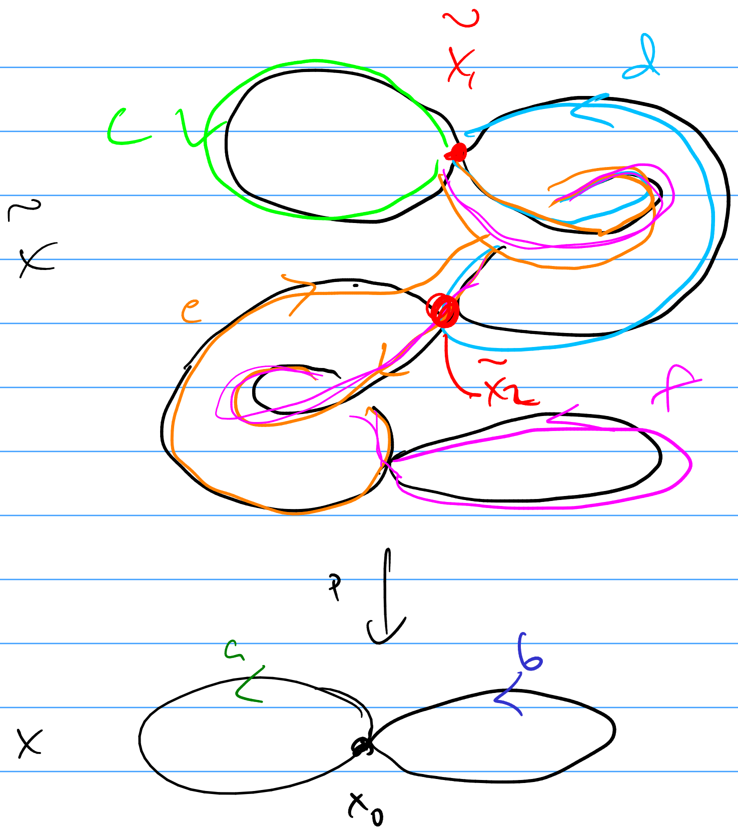
then $[\tilde{\gamma} \cdot \beta \cdot \tilde{\gamma}^{-1}] \in \pi_1(\tilde{X}, \tilde{x}_0)$

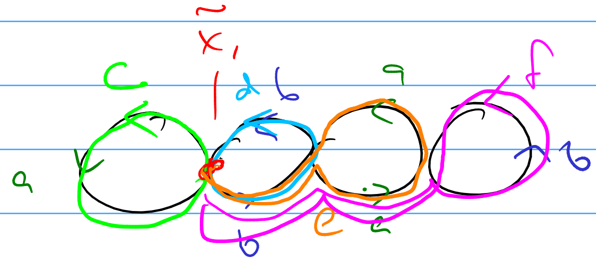
and $p_*[\tilde{\gamma} \cdot \beta \cdot \tilde{\gamma}^{-1}] = [\gamma]\, p_*[\beta]\, [\gamma]^{-1}$

if $\pi_1(\tilde{X}) \triangleleft \pi_1(X)$ is a normal subgroup, basepoint upstairs doesn't matter.

$X = S^1 \vee S^1$

$\tilde{X} \simeq S^1 \vee S^1 \vee S^1 \vee S^1$



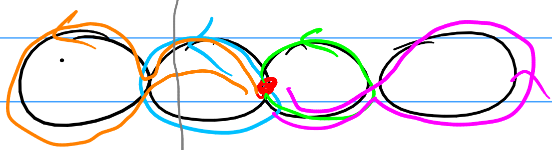


$\pi_1(\tilde{X}, \tilde{x}_1) = \mathbb{Z}_c + \mathbb{Z}_d + \mathbb{Z}_e + \mathbb{Z}_f$

$p_*(\tilde{X}, \tilde{x}_1) = \langle a, b, ba^2b^{-1}, baba^{-1}b^{-1} \rangle \subset \langle a, b \rangle$

conjugate by b

different base point:



$p_* \pi_1(\tilde{X}, \tilde{x}_2) = \langle b^2, a^2, bab^{-1}, aba^{-1} \rangle$

Lifting criterion:

Let $f : Y \longrightarrow X$ be any map
$\quad y_0 \longmapsto x_0$

let $p : \tilde{X} \longrightarrow X$ be a covering
$\quad \tilde{x}_0 \longmapsto x_0$

then $\exists\ \tilde{f} : Y \longrightarrow \tilde{X}$ with $\tilde{f}(y_0) = \tilde{x}_0$ and $p \circ \tilde{f} = f$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

iff $f_* \pi_1(Y, y_0) \subset p_* \pi_1(\tilde{X}, \tilde{x}_0)$

Easy direction: if $f = p \circ \tilde{f}$

then $f_* \pi_1(Y, y_0) = p_*\left(\tilde{f}_*\left(\pi_1(Y, y_0)\right)\right)$

Hard direction: next time

$\subset p_*\left(\pi_1(\tilde{X}, \tilde{x}_0)\right)$

Cor. Any $f : S^n \longrightarrow S^1$ is nullhomotopic. $n \geq 2$

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ S^n \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd}$$

$p(x) = e^{2\pi i x}$

$\operatorname{im} f_* = \{1\} = \operatorname{im} p_*$

and $\mathbb{R}$ is contractible.