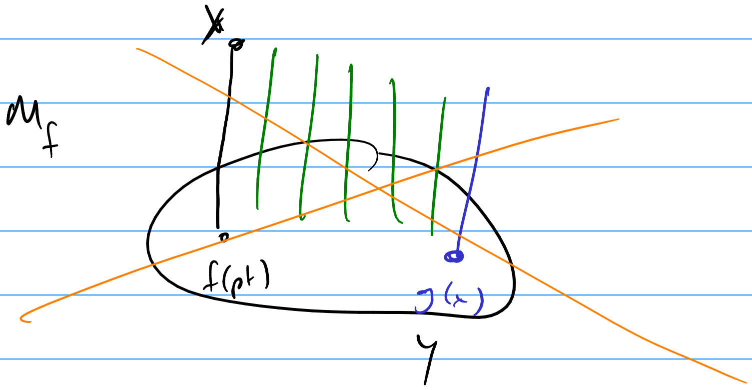
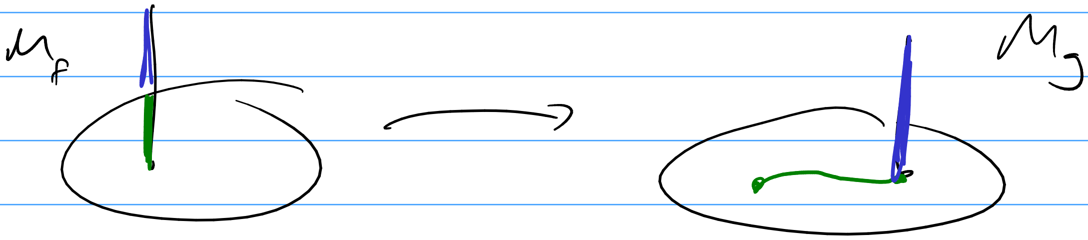
Exam:

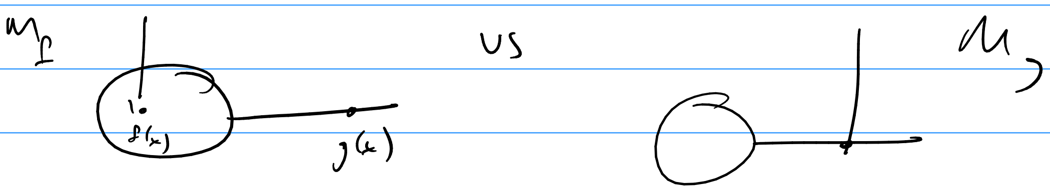
#1 went well

#2 — half perfect, some almost perfect,
many did the following:
if X = point



don't deform M_f to M_g as subspaces of some ambient space

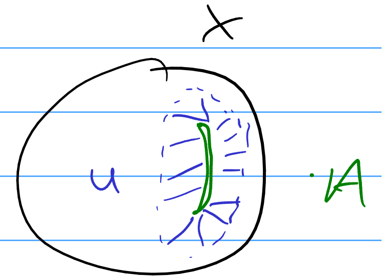




clearer if $\nexists$ homeo: $Y \to Y$
that turns f into g...

application: suppose $A \subset X$ is contractible

want to say $X \to X/A$ is a htpy equiv.



not true in gen.
if $\exists$ nbd $U \supset A$
and a map $f : \partial \bar{U} \to A$
such that $\bar{U} \cong M_f$
then you can prove $X \to X/A$ is a htpy equiv
using this problem as key ingredient.

Covering Spaces

this week: everything path-conn.
+ loc. path-conn.

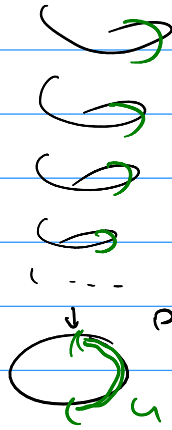
Def. a <u>covering space</u> (or just <u>covering</u>
or <u>cover</u>)
is a map $p : \tilde{X} \to X$
such that every $x \in X$ has a nbd U
for which $p^{-1}(U) \cong U \times$ discrete space
and $p : p^{-1}(U) \to U$

$$p^{-1}(U) \cong U \times \text{discr.} \xrightarrow{\pi_1} U$$

Examples:

1. $\mathbb{R} \longrightarrow S^1$

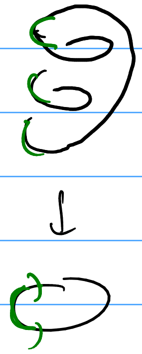
   $x \longmapsto e^{2\pi i x}$



$p^{-1}(U) \cong U \times \mathbb{Z}$

2. $S^1 \longrightarrow S^1$

   $z \longmapsto z^k$



$\pi^{-1}(U) = U \times k$ points

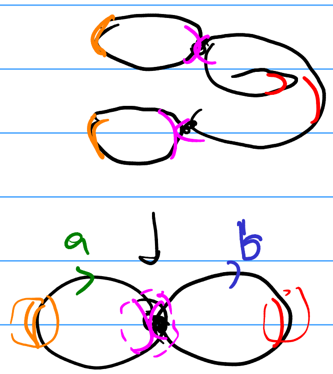
(k-to-1 cover)

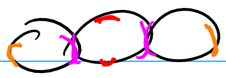
3. $S^n \longrightarrow \mathbb{RP}^n$ 2:1 cover

   $v \longmapsto \mathbb{R}v \subset \mathbb{R}^{n+1}$

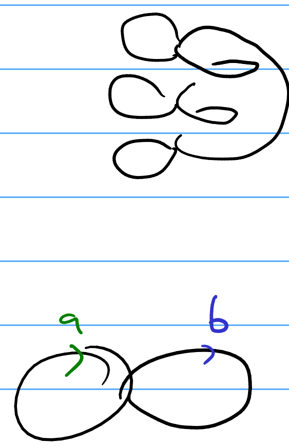
4. three covers of $S^1 \vee S^1$

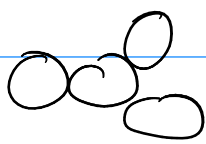
(a)

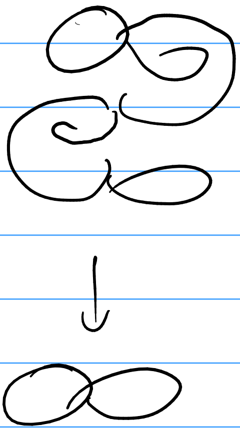


($\tilde{X}$ = )

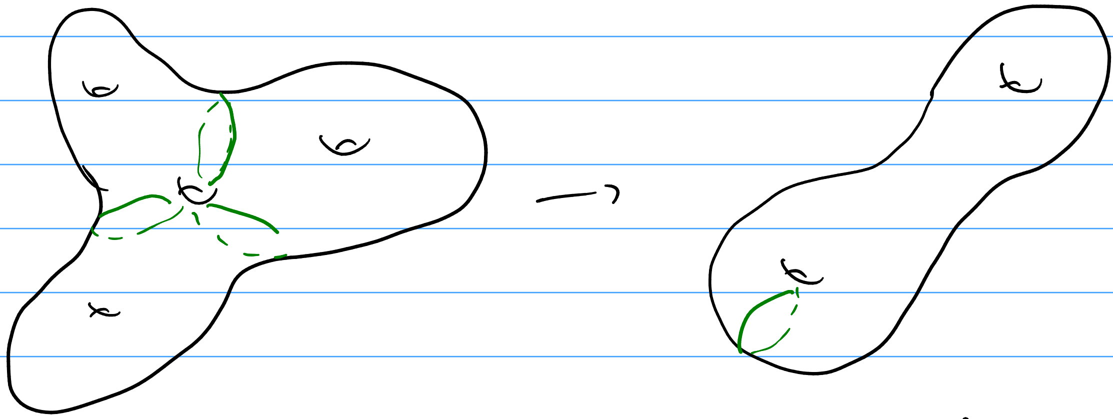
(b)



($\tilde{X}$  $\simeq (S^1)^{\vee 4}$)

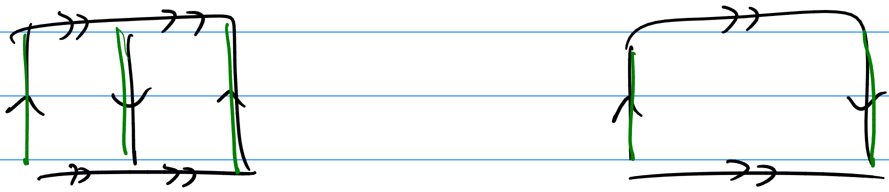
(c)  ($\tilde{X}$ = )

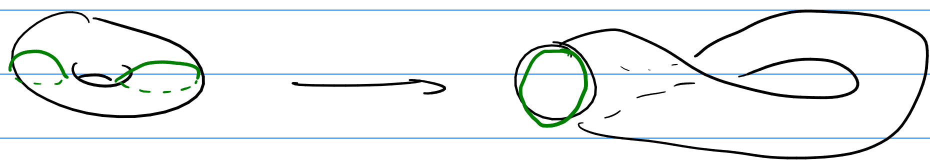
5. 4-holed torus $\xrightarrow{3:1}$ 2-holed torus,



like #4(b)

6. torus $\xrightarrow{2:1}$ klein bottle





let $p: \tilde{X} \to X$ be a cover

let $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$

lemma:

path lifting: $\forall\, \gamma : I \to X$
with $\gamma(0) = x_0$

$\exists!\, \tilde{\gamma} : I \to \tilde{X}$
with $\tilde{\gamma}(0) = \tilde{x}_0$
and $p \circ \tilde{\gamma} = \gamma$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ I \arrow[r, "\gamma"'] \arrow[ur, dashed, "\tilde{\gamma}"] & X \end{tikzcd}$$

homotopy lifting: if $\gamma_0, \gamma_1 : I \to X$

with $\gamma_0(0) = \gamma_1(0) = x_0$
$\gamma_0(1) = \gamma_1(1)$

are homotopic rel. endpoints

then $\tilde{\gamma}_0, \tilde{\gamma}_1 : I \to \tilde{X}$
are homotopic rel. end points.

proofs the <u>same</u> as for $p: \mathbb{R} \to S^1$ earlier.

Prop $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$

is injective.

Pf: suppose $\tilde{\gamma}_0, \tilde{\gamma}_1 : I \longrightarrow \tilde{X}$

loops based at $\tilde{x}_0$

let $\gamma_0 = p \circ \tilde{\gamma}_0 : I \longrightarrow X$

$\gamma_1 = p \circ \tilde{\gamma}_1$

these are loops based at x_0

if $p_*[\tilde{\gamma}_0] = p_*[\tilde{\gamma}_1]$ in $\pi_1(X)$

then $\gamma_0 \simeq \gamma_1$ rel. endpoints

so $\tilde{\gamma}_0 \simeq \tilde{\gamma}_1$ " "

so $[\tilde{\gamma}_0] = [\tilde{\gamma}_1]$ in $\pi_1(\tilde{X})$

Prop $\pi_1(X, x_0)$ acts on the fiber $p^{-1}(x_0)$

stab. of $\tilde{x}_0$ is $p_* \pi_1(\tilde{X}, \tilde{x}_0)$

idea: $\gamma \longmapsto \tilde{\gamma}(1)$

more next time...