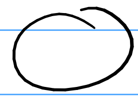
Some Knots via van Kampen's Thm.

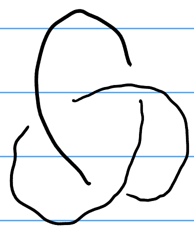
A **knot** is an embedding $f: S^1 \hookrightarrow \mathbb{R}^3$

or better: $f: S^1 \hookrightarrow S^3$.

Example: the un-knot



or the trefoil knot



An **isotopy** between two knots f_0 and f_1

is a homotopy where f_t is an embedding $\forall t$.

One invariant to distinguish between knots:

$$\pi_1(S^3 \setminus f(S^1))$$

today: $\pi_1(S^3 \setminus \text{unknot}) = \mathbb{Z}$

$$\pi_1(S^3 \setminus \text{trefoil}) = \langle a, b \mid a^2 = b^3 \rangle$$

Different: second group surjects onto

$$D_3 = \langle r, s \mid s^2 = r^3 = 1 \mid rs = sr^{-1} \rangle$$

which is not Abelian.

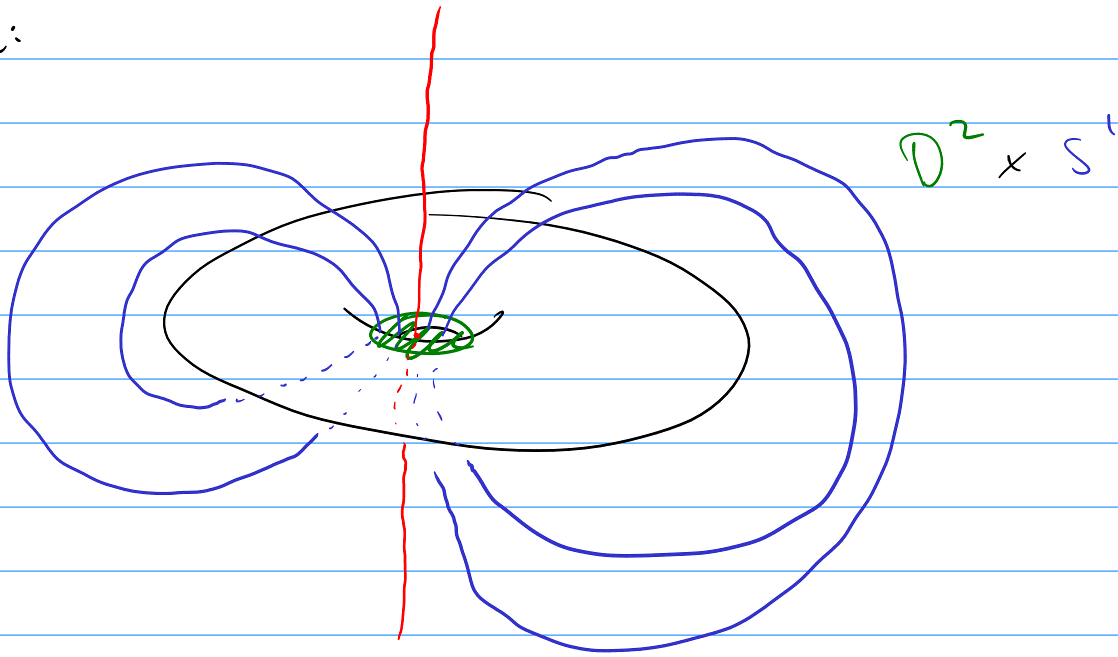
Consider solid torus

 $= S^1 \times D^2 \subset \mathbb{R}^3 \subset S^3$

What is the complement?

Another solid torus...

Picture:



Equations: $S^3 = \{ (z,w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1 \}$

two solid tori: (1) $|z|^2 \le 1/2$ and $|w|^2 \ge 1/2$

(2) $|z|^2 \ge 1/2$ and $|w|^2 \le 1/2$

(1) $\cong D^2 \times S^1$ via

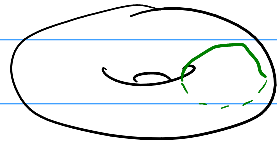
$$(z,w) \hookrightarrow \left(\sqrt{2}\, z, \frac{w}{|w|} \right)$$

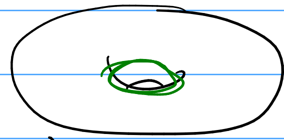
(2) is similar.

Analogy: $S^2 = D^2 \times S^0 \cup S^1 \times D^1$ intersecting along $S^1 \times S^0$

$z = \pm 1/\sqrt{2}$

$=$

if $T = D^2 \times S^1$ a solid torus 

$T' = D^2 \times S^1$ another copy 

glue boundaries via $(z, w) \sim (w', z')$ $\rightsquigarrow$ get S^3

if instead we glued via $(z, w) \sim (z', w')$

$\rightsquigarrow$ get $S^2 \times S^1 \neq S^3$

Glue via another homeo. $\partial T \cong \partial T'$

$\rightsquigarrow$ get some other 3-manifold.

Thm: every compact 3-manifold is
2 solid n-holed tori
glued via an automorphism of the boundary.

So $\pi_1(S^3 \setminus \text{unknot}) = \pi_1(S^3 \setminus \text{solid torus})$

$= \pi_1(\text{other solid torus})$

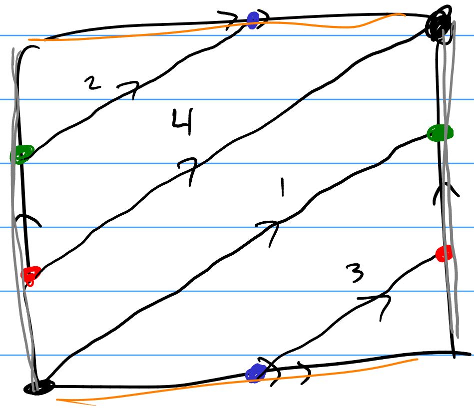
$= \mathbb{Z}.$

$\pi_1(S^3 \setminus \text{trefoil} \ldots)$

Consider torus (surface)

$S^1 \times S^1 \subset S^3$

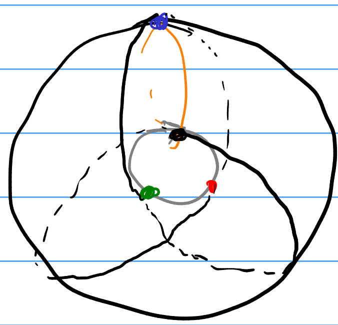
=

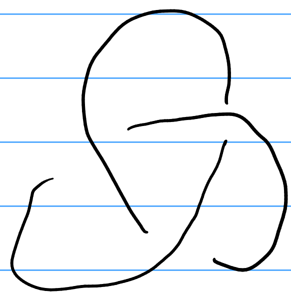


take line of slope $2/3$.

this gives the trefoil knot $S^1 \subset S^1 \times S^1 \subset S^3$

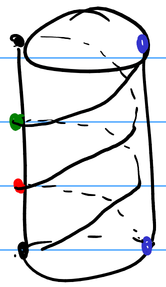
top view:



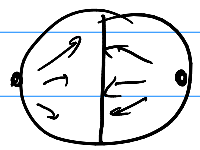


Use Van Kampen to compute $\pi_1(S^3 \setminus \text{trefoil})$

glue $\uparrow$



cross-section



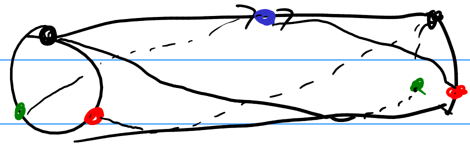
solid cyl $\longleftrightarrow$ solid torus inside earlier picture of $\odot \subset S^3$

solid cyl. $\setminus K \simeq$ Möbius strip with 3 half-twists

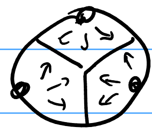
$U =$ 

(but 3 twists)

glue $\twoheadrightarrow$



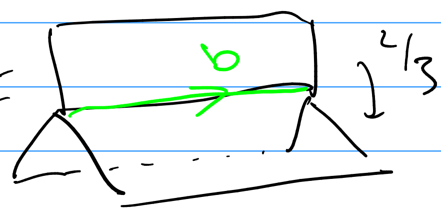
cross-section is



this solid cyl $\longleftrightarrow$ outside of $\odot \subset S^3$

this solid cyl. $\setminus K$

$\simeq$ "3-pronged Möbius strip w/ $2/3$ twist"

$V =$ 

$\partial U = \partial V = S^1$

$S^3 \setminus K \simeq U$ and V glued along boundary.

$\pi_1(U) = \mathbb{Z}$ let a = generator. boundary circle $= a^2$

$\pi_1(V) = \mathbb{Z}$ let b = generator. boundary circle $= b^3$

take appropriate thickenings..

van Kampen says $\pi_1(S^3 \setminus K) = \langle a, b \mid a^2 = b^3 \rangle$.

in lieu of $2/3$, could take m/n with $\gcd(m,n) = 1$

"torus knots"