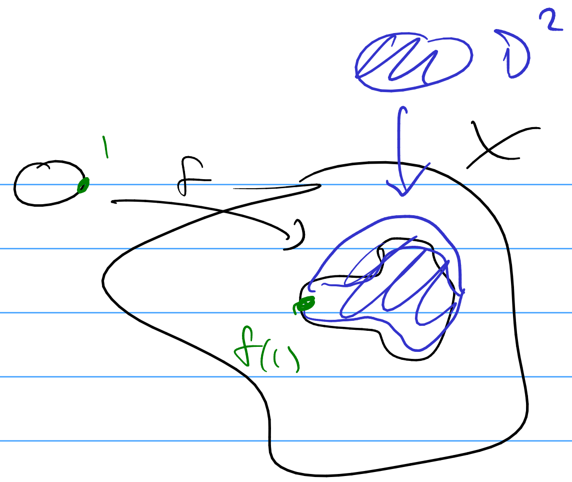
Attaching Discs



let X be a space,

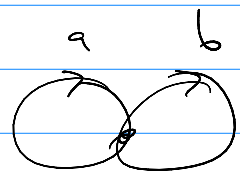
$$f : S^1 \longrightarrow X$$

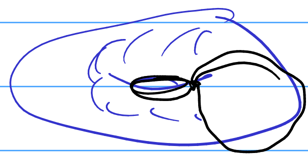
we can attach D^2 along f:

disjoint union

$$X \sqcup D^2 \Big/ p \in \partial D^2 = S^1 \sim f(p) \in X$$

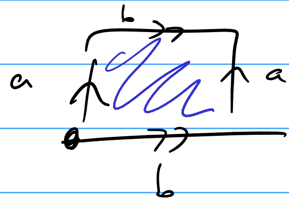
example: $X = S^1 \vee S^1$





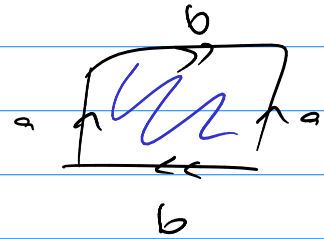
attach a disc along $a b a^{-1} b^{-1}$

$\rightsquigarrow$ get torus



attach a disc along $a b a^{-1} b$

$\rightsquigarrow$ get a Klein bottle

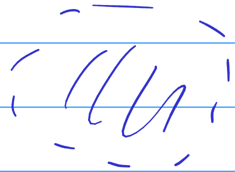


from van Kampen's thm

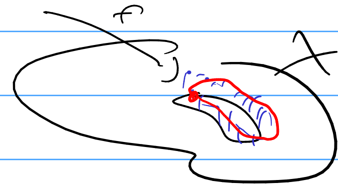
$$\pi_1(X \sqcup_f D^2, f(1)) = \pi_1(X, f(1)) \Big/ [f]$$

Proof:

Let U = interior of disc



$V = X \sqcup_f$ collar of the disc

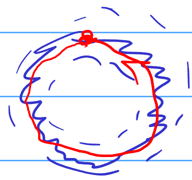


$\pi_1(U) = \{1\}$

V deformation retracts onto X

$\pi_1(V) = \pi_1(X)$

$U \cap V \simeq S^1$



$\pi_1(U \cap V) = \mathbb{Z}$

$$\begin{aligned} \pi_1(U\cap V) &\longrightarrow \pi_1(U) \\ 1 \in \mathbb{Z} &\longmapsto 1 \end{aligned}$$

$$\begin{aligned} \pi_1(U\cap V) &\longrightarrow \pi_1(V) = \pi_1(X) \\ 1 \in \mathbb{Z} &\longmapsto [f] \end{aligned}$$

$$\pi_1(X \sqcup_f D^2) = \pi_1(X) * 1 \Big/ [f] = 1 \quad = \pi_1(X) \Big/ [f]$$

=

examples: $\pi_1(\text{torus}) = \langle a, b \mid \cancel{aba^{-1}b^{-1} = 1} \rangle$

$ab = ba$

$\pi_1(\text{Klein bottle}) = \langle a, b \mid aba^{-1}b = 1 \rangle$

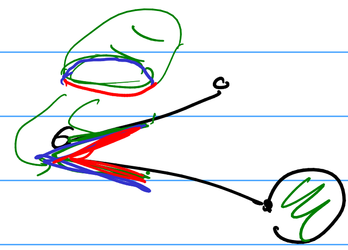
Similarly, given a map $f : S^{n-1} \longrightarrow X$, $n \geq 3$

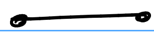
define $$X \underset{f}{\sqcup} D^n = X \cup D^n \Big/ \; p \in \partial D^n = S^{n-1} \sim f(p) \in X$$

Same van Kampen argument
$\rightsquigarrow \pi_1 (X \underset{f}{\cup} D^n) = \pi_1 (X)$

because now $U \cap V \simeq S^{n-1}$
$\pi_1(U \cap V) = 0$ bec. $n \geq 3$.

A CW complex:



start with some points,
attach some D^1's 
attach some D^2's 
etc.

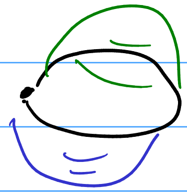
finite dimensional CW complex: stop at some point.
∞-dim'l CW complex: worry about topology on the direct limit.

all our fave spaces
are homeomorphic to CW complexes
or at least homotopy equivalent.

of course they admit many cell structures:

$S^2 =$ D^2 attached to a point

but also



or