$$
\begin{tikzcd}
\text{ker} = \text{normal subgroup generated by } abba \arrow[d] \arrow[ddrr, bend right] & & \text{ker} = \text{norm. subgp. gen'd by } cdcd^{-1} \arrow[d] \\
\mathbb{Z}_a * \mathbb{Z}_b \arrow[rr] \arrow[d] \arrow[drr] & & \mathbb{Z}_c * \mathbb{Z}_d \arrow[d] \\
G = \langle a, b \mid abba = 1 \rangle \arrow[rr, dashed] & & H = \langle c, d \mid cdcd^{-1} = 1 \rangle
\end{tikzcd}
$$

check:

$a \longmapsto ?$

$b \longmapsto ?$

$abba \longmapsto 1$

$\longleftarrow$ and the other way

---

last Wednesday: $\pi_1(S^n) = 0 \quad n \geq 2$

Cor: $\nexists$ homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^n \quad n \geq 3$

because it would induce a homeo

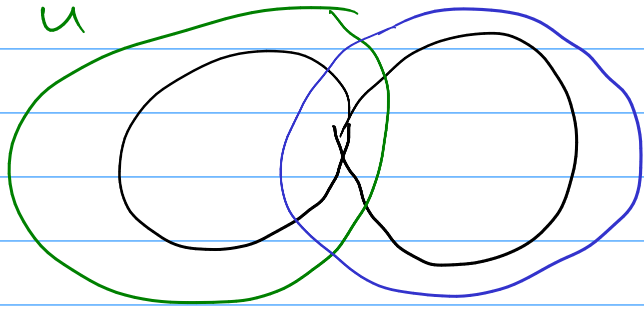
$$\mathbb{R}^2 \setminus 0 \longrightarrow \mathbb{R}^n \setminus f(0)$$

$\hookrightarrow$ htpy equiv to S^{n-1}

and applying π_1,

$$\mathbb{Z} \longrightarrow 0 \quad \text{can't be an iso.}$$

Friday's worksheet: $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$



$U \cap V$ is contractible

apply van Kampen's thm.

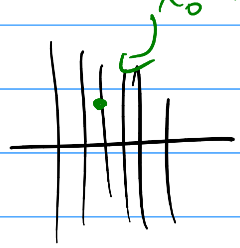
notice: works for any $\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$

as long as basepoint $x_0 \in X$
has a "good" neighborhood
that defo retracts onto x_0

and sim. with $y_0 \in Y$.

happens in any decent example,
but not in

x_0 is not good



$\mathbb{Q} \times \mathbb{R} \cup \mathbb{R} \times \{0\}$

Set-up for van Kampen: $X = U \cup V$

$$\begin{tikzcd} U\cap V \arrow[r, hook, "k"] \arrow[d, hook, "l"'] & U \arrow[d, hook, "i"] \\ V \arrow[r, hook, "j"'] & X \end{tikzcd}$$

get $\pi_1(U) * \pi_1(V) \xrightarrow{i_* * j_*} \pi_1(X)$

let N = normal subgroup gen'd by $k_*[\gamma] * l_*[\gamma]^{-1}$

for $[\gamma] \in \pi_1(U \cap V)$

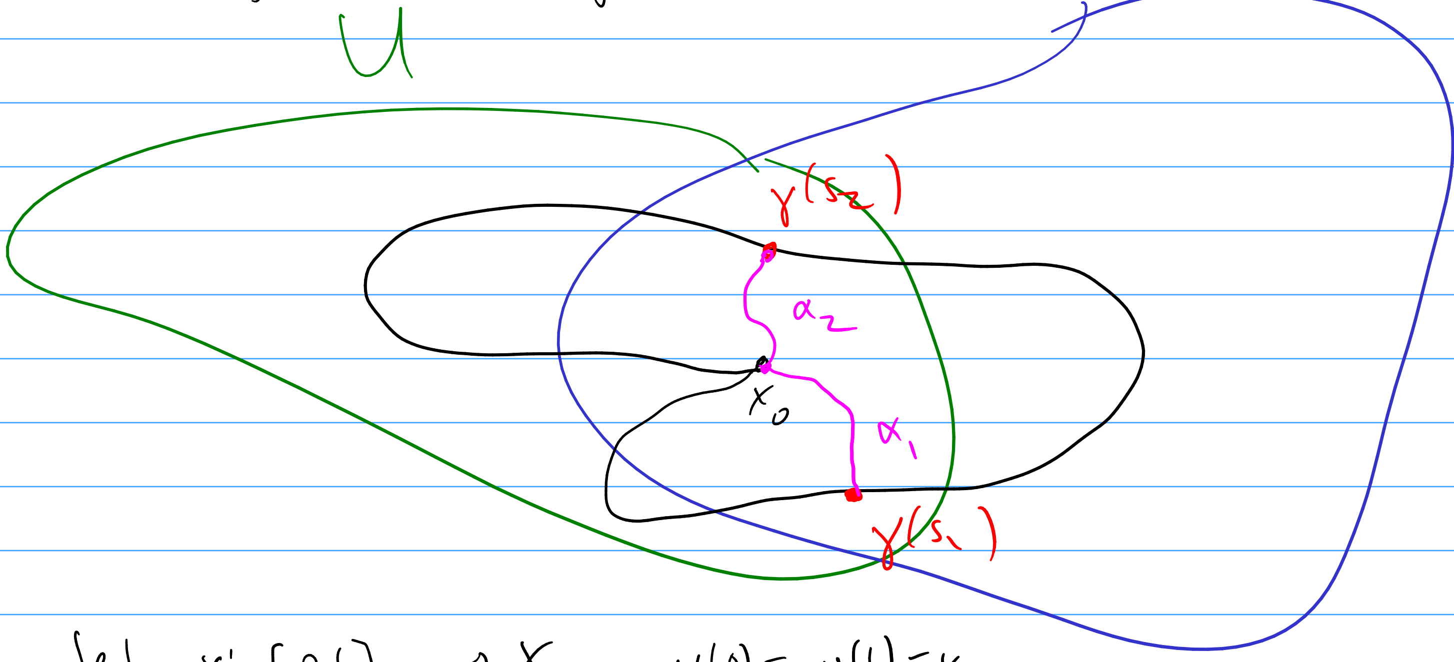
Then $i_* * j_*$ is surjective

+ ker = N

so $\pi_1(X) = \dfrac{\pi_1(U) * \pi_1(V)}{k_*[\gamma] = l_*[\gamma]}$

$\forall [\gamma] \in \pi_1(U\cap V)$

(1) $i_* * j_*$ is surjective.



Let $\gamma : [0,1] \to X$ $\quad \gamma(0) = \gamma(1) = x_0$

by usual compactness argument,
$\exists$ a partition of $[0,1]$

$$0 = s_0 < s_1 < \dots < s_n = 1$$

s.t. $\gamma([s_{i-1}, s_i]) \subset U$ or V $\quad \forall i = 1, \dots, n$

and $\gamma(s_i) \in U \cap V$ $\quad \forall i$

Choose paths α_i in $U \cap V$
from x_0 to $\gamma(s_i)$ $\quad \forall i$

put $\gamma_i = \alpha_{i-1} \cdot \gamma|_{[s_{i-1}, s_i]} \cdot \alpha_i$ $\quad \forall i$

then $[\gamma_i] \in \pi_1(U)$ or $\pi_1(V)$
depending on i.

and $[\gamma] = i_*[\gamma_1] \cdot j_*[\gamma_2] \cdot$ etc. i_* or $j_*[\gamma_n]$

② injectivity: $\ker (i_* * j_*) = N$

suppose $[f_1]*[f_2]*\cdots*[f_m] \in \pi_1(U) * \pi_1(V)$

and $[g_1]*[g_2]*\cdots*[g_n]$ another one,

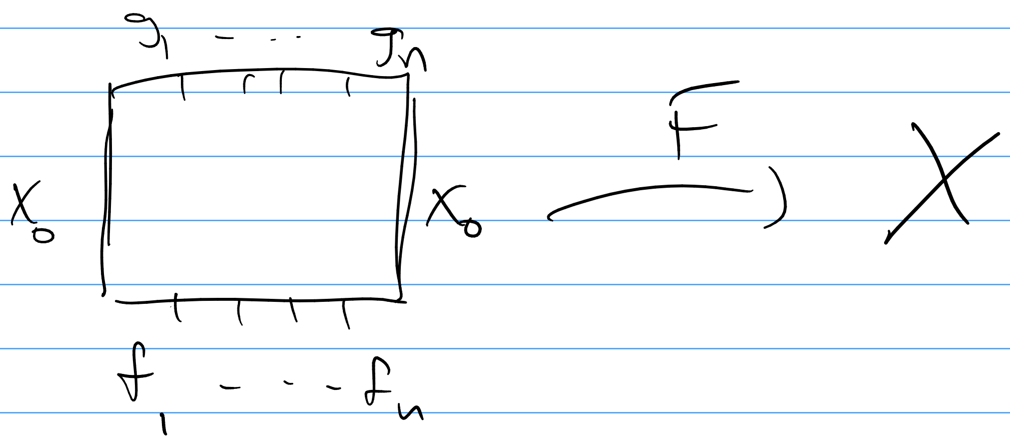
and same image in $\pi_1(X)$.

want: same in $\pi_1(U)*\pi_1(V)/N$

can assume $m=n$

let $F: I \times I \to X$

be a homotopy from $f_1 \cdots f_n$ to $g_1 \cdots g_n$

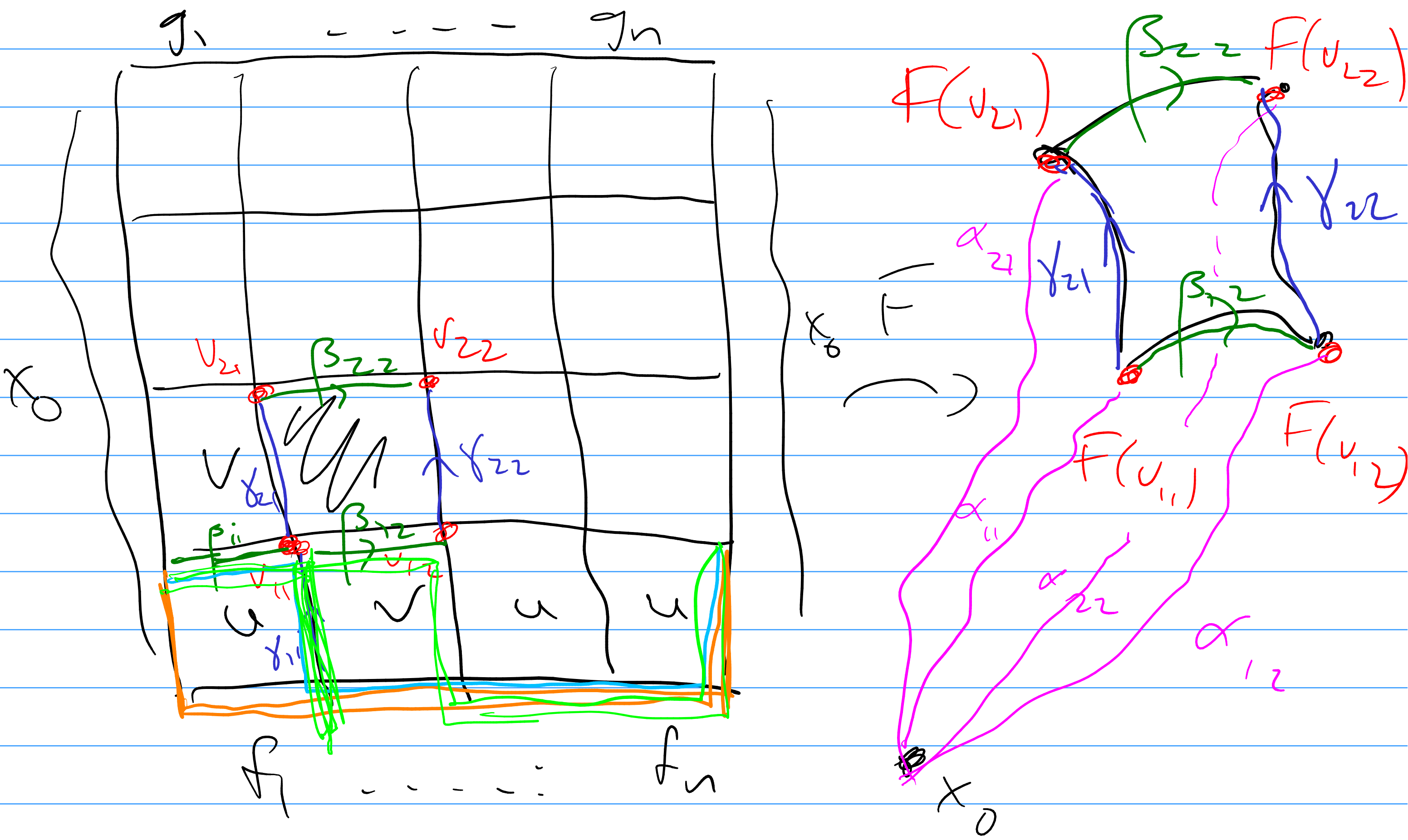


By usual compactness argument,

$\exists\, M >> 0$ s.t. if we subdivide $I \times I$ into M^2 squares

then $F(\text{each one}) \subset U$ or V.

assume $n \mid M$ (picture: $M = n$)



label vertices $v_{11}, v_{12} \ldots$

choose a path α_{ij} from x_0 to $F(v_{ij})$

if $F(v_{ij}) \in U$, make α_{ij} stay in U

— — — — — V — — — — — — — V

let $\beta_{11}, \beta_{12},$ etc $=$ F restr. to horizontal segments

let γ_{11}, γ_{12} etc

$= F|_{\text{vertical segments}}$

First row is $f_1 f_2 \cdots f_n$

Second row gives

$$(\beta_{11}\alpha_{11}^{-1})(\alpha_{11}\beta_{12}\alpha_{12}^{-1}) \cdots (\alpha_{1,n-1}\beta_{1n})$$

both in $\pi_1(U)$ & $\pi_1(V)$

want: same in $\pi_1(U) * \pi_1(V) / N$

$[f_1] * [f_2] * \cdots * [f_n]$

$= [\beta_{11}\alpha_{11}^{-1}][\alpha_{11}\gamma_{11}^{-1}] * [f_2] * \cdots * [f_n]$

$= [\beta_{11}\alpha_{11}^{-1}][\alpha_{11}\gamma_{11}^{-1}]$ in N

$\in \pi_1(U)$ $* [\gamma_{11}\alpha_{11}^{-1}][\alpha_{11}\beta_{12}\alpha_{12}^{-1}][\alpha_{12}\gamma_{12}^{-1}]$

$\in \pi_1(V)$

$= [f_3] * \cdots * [f_n]$

So we can cancel them bec. we've modded out N.

$$= [\beta_{11}\alpha_{11}^{-1}] + [\alpha_{11}\beta_{12}\alpha_{12}^{-1}][a_{12}\gamma_{12}^{-1}] + [f_3] + \dots + [f_n]$$

Keep going square by square, end up with $g_1 \dots g_n$.