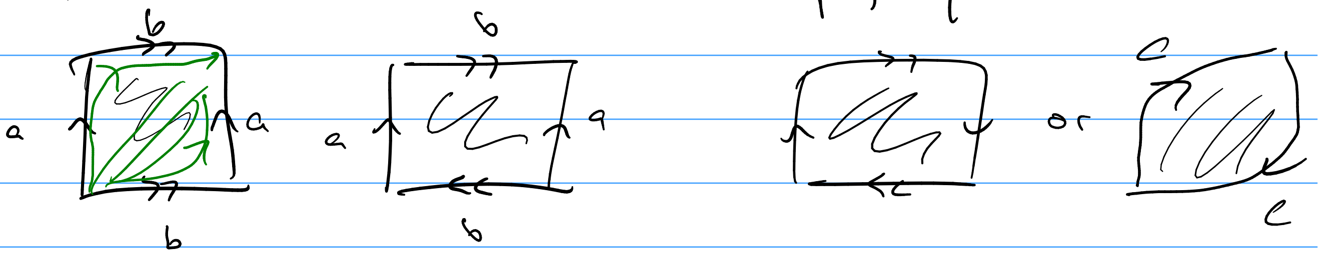
Van Kampen's Theorem

three fav. surfaces:

torus Klein bottle proj. plane



Dream: $\pi_1(\text{torus}) \xleftarrow{\cong} \langle a, b \mid ab = ba \rangle = \mathbb{Z} \times \mathbb{Z}$

$\pi_1(\text{Klein bottle}) \xleftarrow{\cong} \langle a, b \mid ab = b^{-1}a \rangle$

$\pi_1(\text{proj. plane}) \xleftarrow{\cong} \langle c \mid c = c^{-1} \rangle = \mathbb{Z}/2$

easy: maps are surjective
(make $\gamma : I \to$ Klein bottle
be not surjective, like with S^n
then $[\gamma] =$ product of a's and b's)

hard: injective.

Free Products of Groups

Let G, H be groups

The free product $G * H$

is the set of formal strings

$$g_1 * h_1 * g_2 * h_2 * \cdots * g_k * h_k$$

modulo

$$\cdots * g_1 * 1 * g_2 * \cdots$$
$$= \cdots * g_1 g_2 * \cdots$$

multiply by concatenating.

e.g. $(g_1 * h_1 * g_2 * 1)(g_3 * h_3)$

$$= g_1 * h_1 * g_2 g_3 * h_3$$

Example: $\underset{\{a^n\}}{\mathbb{Z}} * \underset{\{b^n\}}{\mathbb{Z}}$ = things like $a^2 b a^{-1} b^{-3} a^5$

mult: $a^2 b \cdot a b^2 = a^2 b a b^2$

$a^2 b \cdot b^{-2} a = a^2 b^{-1} a$

Example 2: $\mathbb{Z}/2 * \mathbb{Z}/2$ = things like $ababa$ $\quad baba$

$$\begin{array}{cc} \mathbb{Z}/2 & \mathbb{Z}/2 \\ / & \backslash \\ \{1,a\} & \{1,b\} \\ a^2=1 & b^2=1 \end{array}$$

mult: $aba \cdot bab = ababab$

$aba \cdot a = ab$

Universal property:

$$\begin{tikzcd}
 & H \arrow[d, hook] \arrow[ddr, bend left, "\psi"] & \\
G \arrow[r, hook] \arrow[drr, bend right, "\varphi"'] & G*H \arrow[dr, dashed, "\exists!"] & \\
 & & K
\end{tikzcd}$$

"coproduct"

$$(\varphi * \psi)(g_1 * h_1 * \cdots * g_k * h_k) = \varphi(g_1)\psi(h_1)\cdots\varphi(g_k)\psi(h_k) \in K.$$

$\varphi * \psi$ is a hom.

Non-example:

$S_3 = \{1, (12), (13), (23), (123), (132)\}$

is generated by $G = \{1, (12)\} \cong \mathbb{Z}/2$

and $H = \{1, (123), (132)\} \cong \mathbb{Z}/3$

map $f: S_3 \to K$ is determined by $f|_G : G \to K$

and $f|_H : H \to K$

but you can't do whatever you want"

$\varphi : S_3 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_3$

$(12) \longmapsto (1,0)$
$(123) \longmapsto (0,0)$

is a well-defined homom.

$(12) \longmapsto (1,0)$
$(123) \longmapsto (0,1)$

$\varphi|_G$ and $\varphi|_H$ are well-defined homs.

but φ is not:

$$(132) = (123)^2 \longmapsto (0,2)$$
$$\neq$$
$$= (12)(123)(12) \longmapsto (2,1)$$

so $S_3 \not\cong \mathbb{Z}_2 \times \mathbb{Z}_3$

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if G is a group and $S \subset G$ a subset,

the normal subgroup generated by S

$= \bigcap(\text{all normal subgroups} \supset S)$

$=$ subgroup gen'd by $gsg^{-1} \mid g \in G, s \in S$

Let $X = U \cup V$ (open sets)

$U, V, U \cap V$ are path conn.

(basepoint $x_0 \in U \cap V$ -- suppress from notation)

$$\begin{tikzcd} U \cap V \arrow[r, hook, "l"] \arrow[d, hook, "k"'] & V \arrow[d, hook, "j"] \\ U \arrow[r, hook, "i"'] & X \end{tikzcd}$$

$$i_* * j_* : \pi_1(U) * \pi_1(V) \to X$$

$$\underset{\text{in } U}{[\alpha_1]} * \underset{\text{in } V}{[\beta_1]} * \cdots \mapsto i_*[\alpha_1]\, j_*[\beta_1] \cdots$$

$$= [(i \circ \alpha_1) \cdot (j \circ \beta_1) \cdots]$$

obvious elements of $\ker(i_* * j_*)$:

for $[\gamma] \in \pi_1(U \cap V)$

$$k_*[\gamma] * l_*[\gamma]^{-1} \longmapsto i_* k_* [\gamma] \cdot j_* l_* [\gamma]^{-1}$$

$= 1$ bec $ik = jl$

Thm (VK): $i_* * j_*$ is surjective

$\ker =$ normal subgroup N gen'd by these

So $\pi_1(X) = \pi_1(U) * \pi_1(V) / N$