Comment on worksheet:

AoC: given a collection of non-empty sets X_y indexed by a set Y,

$\exists$ a choice function
$$c : Y \longrightarrow \bigcup_{y \in Y} X_y$$
s.t. $c(y) \in X_y \quad \forall\, y \in Y$

consider $X = \{ (x,y) \mid y \in Y \quad x \in X_y \}$

and the map $f : X \to Y$
$$f(x,y) = y.$$

Surjective bec. each $X_y \neq \emptyset$
section of f = same as a choice function c

Thm. let
$$\begin{array}{ccc} X \times Y & \xrightarrow{q} & Y \\ p \downarrow & & \\ X & & \end{array}$$

let $x \in X$ and $y \in Y$

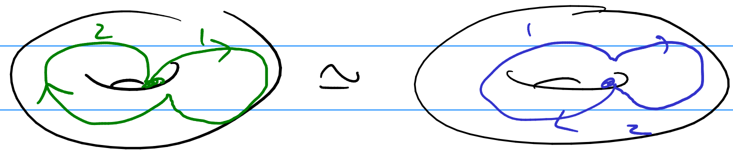
then the map

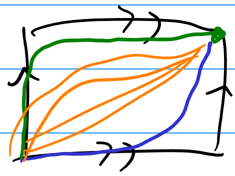
$$\Phi : \pi_1(X \times Y, (x,y)) \longrightarrow \pi_1(X,x) \times \pi_1(Y,y)$$
$$[\gamma] \longmapsto (p_*[\gamma], q_*[\gamma])$$

is an isomorphism.

example: $\pi_1(\text{torus}) = \mathbb{Z} \times \mathbb{Z}$

because torus $= S^1 \times S^1$





Pf. Φ is a group hom bec. p_* and q_* are.

Φ is surjective. given $[\alpha] \in \pi_1(X, x)$
$[\beta] \in \pi_1(Y, y)$

let $\gamma = \alpha \times \beta : I \longrightarrow X \times Y$

that is, $\gamma(s) = (\alpha(s), \beta(s))$

then $\gamma(0) = \gamma(1) = (x, y)$
so $[\gamma] \in \pi_1(X \times Y, (x, y))$

$p \circ \gamma = \alpha \qquad q \circ \gamma = \beta$
so $p_*[\gamma] = [\alpha] \qquad q_*[\gamma] = [\beta]$

$\Phi([\gamma]) = ([\alpha], [\beta])$

Φ is injective: suppose we have $[\gamma_0], [\gamma_1] \in \pi_1(X \times Y)$

with $\Phi([\gamma_0]) = \Phi([\gamma_1])$

then $p_*[\gamma_0] = p_*[\gamma_1]$

let $F : I \times I \to X$ be a htpy rel. endpoints from $p \circ \gamma_0$ to $p \circ \gamma_1$

similarly, let $G : I \times I \to Y$

from $q \circ \gamma_0$ to $q \circ \gamma_1$

now define $H = F \times G : I \times I \to X \times Y$

that is, $H(s,t) = (F(s,t), G(s,t))$

then H is a homotopy rel endpoints from γ_0 to γ_1. $\square$

Theorem: $\pi_1(S^n, \text{south pole}) = 0$ if $n \geq 2$.

Proof: Let $\gamma : I \to S^n$

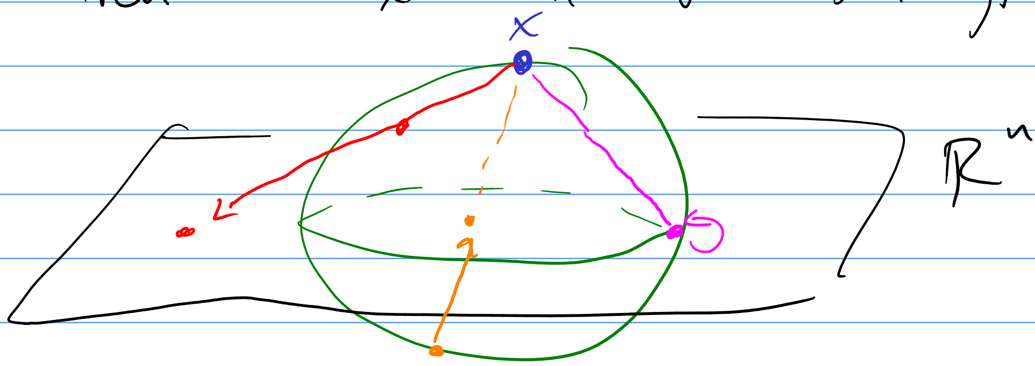
$$\gamma(0) = \gamma(1) = \text{south pole}$$

Want to show $\gamma \simeq$ const path.

if γ is not surjective,

let $x \in S^n \setminus \gamma(I)$

then $S^n \setminus x \cong \mathbb{R}^n$ via stereographic proj:



We can factor

$$\begin{tikzcd} I \arrow[r, "\delta"'] \arrow[rr, bend left, "\gamma"] & S^n \setminus x \arrow[r, hook, "i"'] & S^n \end{tikzcd}$$

$S^n \setminus x \cong \mathbb{R}^n$ is contractible

so $\delta \simeq$ const map

so $i \circ \delta = \gamma \simeq$ const map.

Not done: $\exists$ space-filling curves $\gamma : I \to S^n$.

Worksheet: every cont $\gamma : I \to S^n$ is homotopic to one that's not surjective.