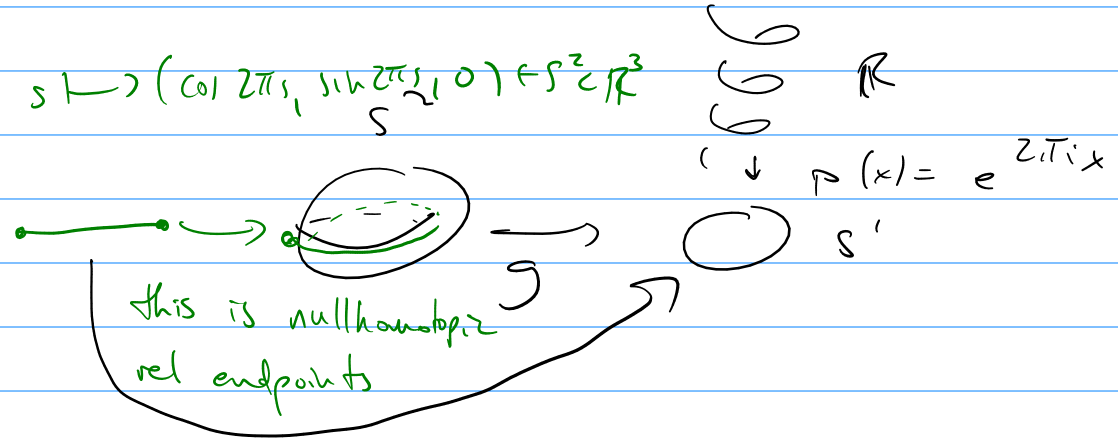
Friday's worksheet:

point is $\nexists$ $g : S^2 \to S^1$
with $g(-x) = -g(x)$

assume $g(1) = 1$, else replace g with $g(x)/g(1)$



$\gamma : I \to S^1$ is nullhomotopic rel. endpoints.

for $s \in [1/2, 1]$,
we have $\gamma(s - 1/2) = -\gamma(s)$
because $g(-x) = -g(x)$

get $\tilde{\gamma} : I \to \mathbb{R}$ $\tilde{\gamma}(0) = 0$ $p \circ \tilde{\gamma} = \gamma$

$p(\tilde{\gamma}(1/2)) = \gamma(1/2) = -\gamma(0) = -1$

so $\tilde{\gamma}(1/2) = n + \frac{1}{2}$ for some $n \in \mathbb{Z}$

for $s \in [1/2, 1]$, $\tilde{\gamma}(s) = \tilde{\gamma}(s - 1/2) + n + \frac{1}{2}$
bec. they agree at $s = 1/2$
and $p(\tilde{\gamma}(s)) = \gamma(s)$ $p(\tilde{\gamma}(s - 1/2) + n + 1/2) = - - \gamma(s)$

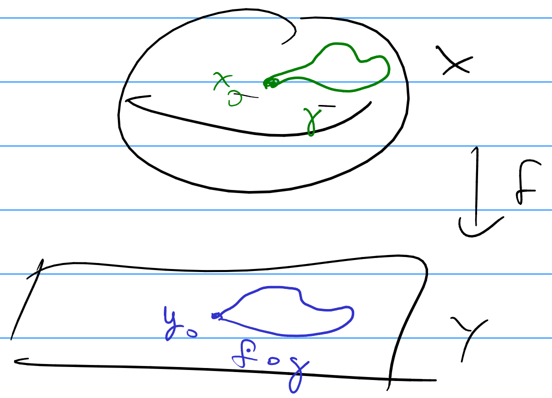
thus $\tilde{\gamma}(1) = 2n+1$
so $[\gamma] \neq 0$ in $\pi_1(S^1, 1)$

Induced maps.

Given $f: (X, x_0) \longrightarrow (Y, y_0)$

define $f_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$

$$[\gamma] \longmapsto [f \circ \gamma]$$



well-defined: if $\gamma_0 \simeq \gamma_1$ rel endpoints

then $f \circ \gamma_0 \simeq f \circ \gamma_1$

group hom: $f \circ (\gamma_1 \cdot \gamma_2) = (f \circ \gamma_1) \circ (f \circ \gamma_2)$.

Example: $f: S^1 \longrightarrow S^1$ $\quad f(1) = 1$

$z \longmapsto z^2$

what is $f_*: \pi_1(S^1, 1) \longrightarrow \pi_1(S^1, 1)$

$\mathbb{Z} \longrightarrow \mathbb{Z}$

$n \longmapsto 2n$

$n \in \mathbb{Z} = \pi_1(S^1, 1)$ represented by

$\gamma: I \longrightarrow S^1 \quad \gamma(s) = e^{2\pi i n s}$

then $f(\gamma(s)) = e^{4\pi i n s}$ represents $2n$

similarly $\quad z \mapsto z^3$ induces $n \mapsto 3n$

$z \mapsto z^{-1}$ induces $n \mapsto -n$

etc.

Functoriality:

Suppose

$$\begin{tikzcd} (X,x_0) \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & (Y,y_0) \arrow[r, "g"] & (Z,z_0) \end{tikzcd}$$

hit it with π_1:

$$\begin{tikzcd} \pi_1(X,x_0) \arrow[r, "f_*"] \arrow[rr, bend right, "(g\circ f)_*"'] & \pi_1(Y,y_0) \arrow[r, "g_*"] & \pi_1(Z,z_0) \end{tikzcd}$$

Is $(g\circ f)_* = g_* \circ f_*$?

Yes: $(g\circ f)_*([\gamma]) = [g\circ f\circ \gamma]$

$= g_*([f\circ\gamma])$

$= g_*(f_*([\gamma]))$

Moreover, $1: (X,x_0) \longrightarrow (X,x_0)$

induces the identity map

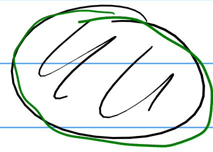
$\pi_1(X,x_0) \longrightarrow \pi_1(X,x_0)$

i.e. $1_* = 1$ $\quad$ [because $1\circ\gamma = \gamma$]

Second perspective on Brouwer fixed pt thm:

If we had $f: D^2 \to D^2$ with $f(x) \neq x \ \forall x$

we constructed $g: D^2 \to S^1$



with $g|_{S^1} = \text{identity}.$

let $i: S^1 \hookrightarrow D^2$ be the inclusion

$$\begin{tikzcd}
(S^1, 1) \arrow[r, "i"] \arrow[rr, bend right, "1"'] & (D^2, 1) \arrow[r, "g"] & (S^1, 1)
\end{tikzcd}$$

Hit it with π_1

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "i_*"] \arrow[rr, bend right, "1"'] & 0 \arrow[r, "g_*"] & \mathbb{Z}
\end{tikzcd}$$

we have $(g \circ i)_* = 1_* = 1$

but $g_* \circ i_* = 0$! contradiction.

Homotopy Invariance

Prop. if $f, g : (X, x_0) \longrightarrow (Y, y_0)$

are homotopic rel. basepoint

then $f_*, g_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$

are equal.

Pf for $[\gamma] \in \pi_1(X, x_0)$,

have $f \circ \gamma \simeq g \circ \gamma$ rel endpoints

so $f_*[\gamma] = g_*[\gamma]$. $\boxtimes$

Cor. if $f : (X, x_0) \longrightarrow (Y, y_0)$ is a htpy equiv. rel basepoint

then f_* is an isomorphism on π_1.

Pf let $g : (Y, y_0) \longrightarrow (X, x_0)$

be a htpy inverse,

i.e. $g \circ f \simeq 1_X$ $f \circ g \simeq 1_Y$ rel. basepoints.

then $g_* \circ f_* = (g \circ f)_* = 1_* = 1$

similarly $f_* \circ g_* = 1$

so f_* is an iso with inverse g_*. $\boxtimes$

Example: $\pi_1(\mathbb{R}^2 \setminus 0) = \pi_1(S^1) = \mathbb{Z}$