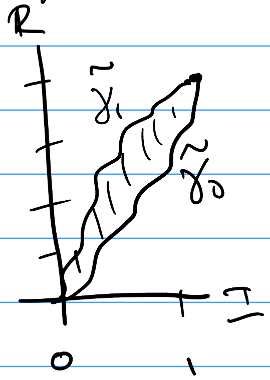
last worksheet

$\gamma_1, \gamma_2 : I \to S^1$

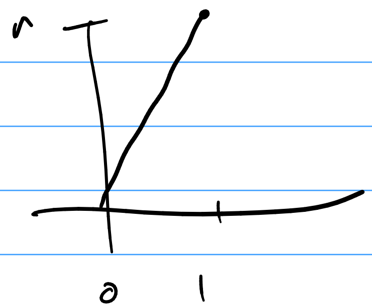
if $\tilde{\gamma}_1(1) = \tilde{\gamma}_2(1)$ then $\gamma_1 \simeq \gamma_2$ rel. endpoints.



$F : \tilde{\gamma}_1 \simeq \tilde{\gamma}_2$ straight line

$p \circ F : \gamma_1 \simeq \gamma_2$

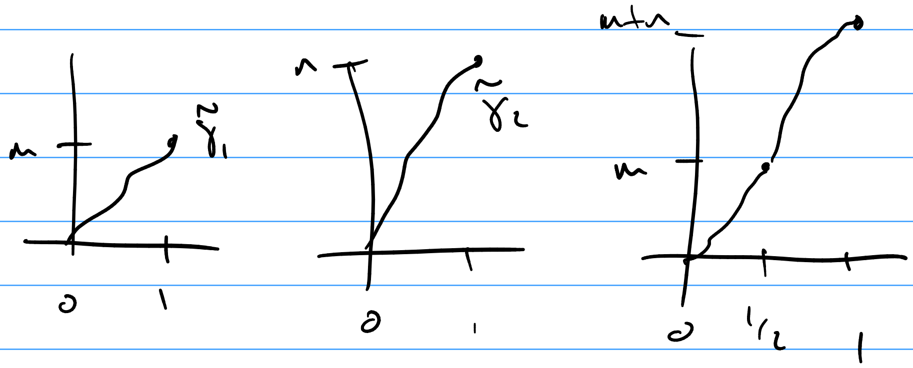
Surjective:



$\tilde{\gamma}(s) = ns$

$\gamma := p \circ \tilde{\gamma}$

group hom.



First Fruits of $\pi_1(S^1) = \mathbb{Z}$.

Lemma: the map $\begin{array}{c} S^1 \longrightarrow S^1 \\ z \longmapsto z^n \end{array}$

is not null homotopic. represents $n \in \mathbb{Z} = \pi_1(S^1, 1)$

Pf: Homework 3.

Fundamental Theorem of Algebra:

every polynomial

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n \qquad a_i \in \mathbb{C} \qquad n \geq 1$$

has a root in $\mathbb{C}$.

Proof: Suppose $p(z) \neq 0$ for $|z| \leq 1$

Then the map $f: S^1 \longrightarrow S^1$

$$z \longmapsto p(z)/|p(z)|$$

extends to a map $D^2 \longrightarrow S^1$

so f is null homotopic.

Suppose $p(z) \neq 0$ for $|z| \geq 1$

define $F : S^1 \times I \longrightarrow S^1$ by

$$F(z,t) = \frac{t^n p(z/t)}{|\text{same}|} = \frac{z^n + a_1 t z^{n-1} + a_2 t^2 z^{n-2} + \cdots + a_n t^n}{|\text{same}|}$$

$F(z,0) = z^n$ bec. $z \in S^1$

$F(z,1) = f(z) = \dfrac{p(z)}{|p(z)|}$

so $z^n \simeq f \simeq$ const map

so we can't have $z \neq 0$ $\square$

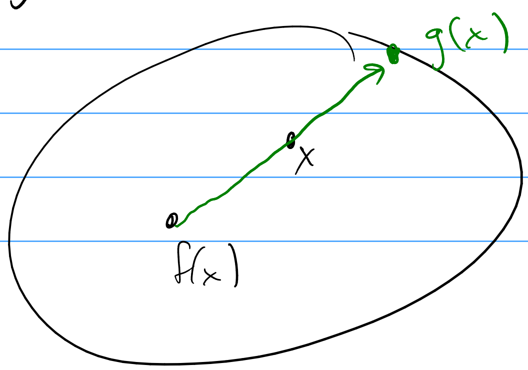
Brouwer Fixed Point Theorem:

Every map $D^2 \xrightarrow{f} D^2$ has a fixed point
i.e. $\exists\ x \in D^2$ with $f(x) = x$.

(Notice: it's not true of the open unit disc $\cong \mathbb{R}^2$.)

Pf Suppose $\exists\ f : D^2 \to D^2$ with no fixed point.

Define $g : D^2 \to S^1$ as follows:



notice: if $x \in S^1$ then $g(x) = x$

So the identity map $S^1 \to S^1$

extends to the disc

So it's nullhomotopic.

But that contradicts the lemma

with $n=1$. $\square$

Worksheet: Borsuk – Ulam thm