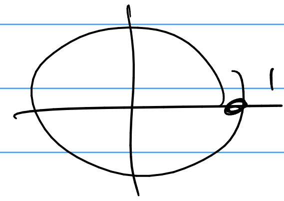
π_1(circle), continued.

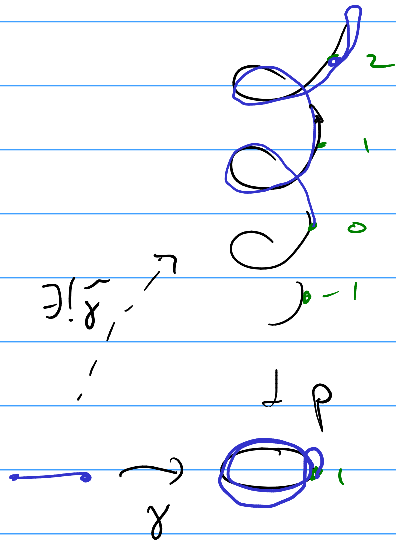
$S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$



proving that $\pi_1(S^1, 1) = \mathbb{Z}$

consider $p: \mathbb{R} \longrightarrow S^1$

$x \longmapsto e^{2\pi i x}$



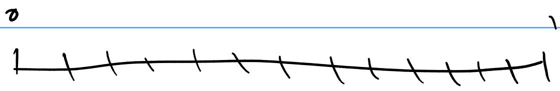
plan from last time:

① $\forall$ $\gamma: I \longrightarrow S^1$
with $\gamma(0) = \gamma(1) = 1$

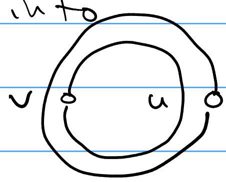
$\exists!$ $\tilde{\gamma}: I \longrightarrow \mathbb{R}$
with $\tilde{\gamma}(0) = 0$ and $p \circ \tilde{\gamma} = \gamma$

proved it on worksheet last time:

outline: break $[0,1]$ into k intervals



s.t. γ takes each one into
either $U = S^1 \setminus \{-1\}$
or $V = S^1 \setminus \{1\}$



lift one interval at a time

because $p^{-1}(U) = \mathbb{Z}$ disjoint copies of U
$p^{-1}(V) = $ - - — - — — . V

observe: if $\alpha, \beta: I \longrightarrow \mathbb{R}$
satisfy $\alpha(0) = \beta(0)$ and $p \circ \alpha = p \circ \beta$
then $\alpha = \beta$.

② Lemma (homotopy lifting)

let $\gamma_0, \gamma_1 : I \to S^1$

with $\gamma_0(0) = \gamma_1(0) = \gamma_0(1) = \gamma_1(1) = 1$

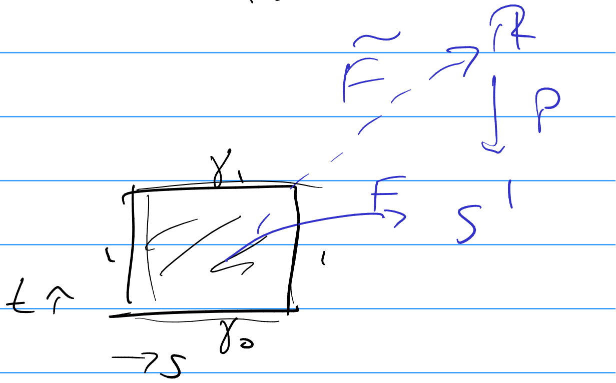
if $\gamma_0 \simeq \gamma_1$ rel. endpoints

then $\tilde{\gamma}_0 \simeq \tilde{\gamma}_1$ rel. endpoints

and in particular $\tilde{\gamma}_0(1) = \tilde{\gamma}_1(1)$.

Proof. let $F : I \times I \to S^1$ be a homotopy rel. endpoints

so $\left.\begin{array}{l} F(s,0) = \gamma_0(s) \\ F(s,1) = \gamma_1(s) \end{array}\right| \forall s \in I \qquad \left.\begin{array}{l} F(0,t) = 1 \\ F(1,t) = 1 \end{array}\right\} \forall t \in I$

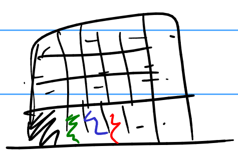


Then $\exists! \ \tilde{F} : I \times I \to \mathbb{R}$

with $\tilde{F}(0,0) = 0$,

and $p \circ \tilde{F} = F$

similar to what we did with path lifting, but now



$F(\text{each little rectangle}) \subset U$ or V

lift this first — then lift this

Now to show that $\tilde{F}$ is a htpy rel. endpoints?

claim 1: $\tilde{F}(s,0) = \tilde{\gamma}_0(s) \quad \forall s \in I$

pf: as s varies, this gives two paths in $\mathbb{R}$

they agree at $s=0$

and $p(\tilde{F}(s,0)) = F(s,0) = \gamma_0(s) = p(\tilde{\gamma}_0(s))$

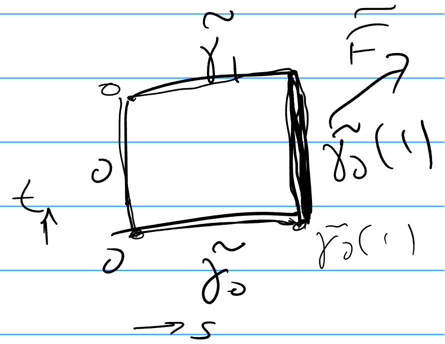
so they agree $\forall s$.

claim 2: $\tilde{F}(0,t) = \underline{\underline{0}} \quad \forall t \in I$

two paths in $\mathbb{R}$

agree at $t=0$

$p(\tilde{F}(0,t)) = F(0,t) = 1 = p(0)$



claim 3: $\tilde{F}(s,1) = \tilde{\gamma}_1(s) \quad \forall s \in I$

two paths agree at $s=0$ by prev. claim.

and $p(\tilde{F}(s,1)) = F(s,1) = \gamma_1(s) = p(\tilde{\gamma}_1(s))$

claim 4: $\tilde{F}(1,t) = \tilde{\gamma}_0(1) \quad \forall t$

agree at $t=0$ by claim 1 above

$p(\tilde{F}(1,t)) = F(1,t) = 1 = \gamma_0(1) = p(\tilde{\gamma}_0(1))$

now claims 3+4 give

$$\tilde{\gamma}_1(1) = \tilde{F}(1,1) = \tilde{\gamma}_0(1)$$ ☒

So the map
$$\varphi : \pi_1(S^1,1) \longrightarrow \mathbb{Z}$$
$$[\gamma] \longmapsto \tilde{\gamma}(1) \in p^{-1}(1) = \mathbb{Z} \subset \mathbb{R}$$

is well-defined.

All downhill from here: worksheet.