Last worksheet:

$$\pi_1(X,x) = \{\text{maps } \gamma: I \longrightarrow X \mid \underbrace{\gamma(0) = \gamma(1) = x}\} \Big/ \text{htpy rel. endpoints}$$

$$\Big\uparrow \qquad \gamma = \ell \circ f \;\; \uparrow \overline{\ell}$$

$$\{\text{maps } \ell: S^1 \longrightarrow X \mid \underbrace{\ell(1) = x}\} \Big/ \text{htpy rel basepoint}$$

$$\begin{tikzcd} I \arrow[d, "f"'] \arrow[dr, "\gamma"] & \\ S^1 \arrow[r, "\ell"] & X \end{tikzcd}$$

$f(s) = e^{2\pi i s}$ it's a quotient map

$f(0) = f(1) = 1$

well-defined: if $\ell_0 \simeq \ell_1$ then $\gamma_0 = \ell_0 \circ f$
$\simeq \ell_1 \circ f = \gamma_1$

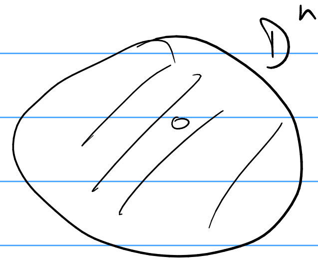
surjective: given γ $\exists$ ℓ
uses univ. prop of quot top.

injective: if $\gamma_0 \simeq \gamma_1$ rel endpoints,
is $\ell_0 \simeq \ell_1$ rel base point?

take htpy
$$\begin{tikzcd} I \times I \arrow[r] \arrow[d, "f \times 1"'] & X \\ S^1 \times I \arrow[ur] & \end{tikzcd}$$

Fundamental Group of the Circle.

def of π_1 is complicated.
how do we compute?



simple example: $\pi_1(D^n, 0) = 0$

Pf. let $\gamma: I \to D^n$ be a loop based at 0.

define $F: I \times I \to D^n$
$(s,t) \mapsto t\gamma(s)$

straight-line htpy from $F(-,0) =$ const path at 0
to $F(-,1) = \gamma$

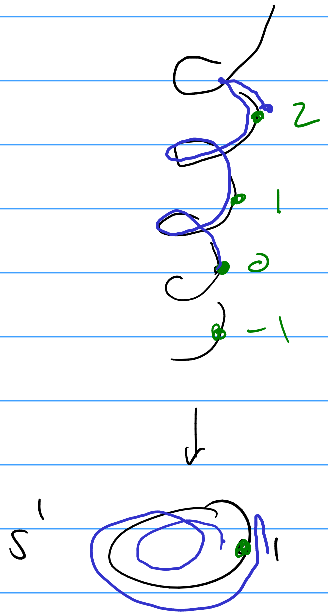
notice $F(0,t) = F(1,t) = 0 \quad \forall t$
so it's a htpy rel endpoints.

thus $\pi_1(D^n, 0) = \{\text{loops}\}/\text{htpy}$
has only one element. $\boxtimes$

$\pi_1(\mathbb{R}^n, 0) = 0$. same proof.

Theorem: $\pi_1(S^1, 1) = \mathbb{Z}$

Idea: consider
$$p: \mathbb{R} \to S^1$$
$$x \mapsto e^{2\pi i x}$$



given a path $\gamma: I \to S^1$ $\quad \gamma(0) = \gamma(1) = 1$

we will lift it to a path $\tilde{\gamma}: I \to \mathbb{R}$ starting at 0

i.e. $p \circ \tilde{\gamma} = \gamma$ and $\hat{\gamma}(0) = 0$

then $p(\tilde{\gamma}(1)) = \gamma(1) = 1$

so $\tilde{\gamma}(1) \in p^{-1}(1) = \mathbb{Z} \subset \mathbb{R}$

we'll define
$$\varphi: \pi_1(S^1, 1) \to \mathbb{Z}$$
$$[\gamma] \mapsto \tilde{\gamma}(1)$$

plan: ① $\forall \gamma \; \exists! \; \tilde{\gamma}$

② φ is well-defined:

$$\gamma \simeq \gamma' \implies \tilde{\gamma}(1) = \tilde{\gamma}'(1)$$

③ φ is injective:

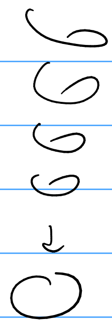
" $\Longleftarrow$ "

④ φ is surjective

⑤ φ is a group hom.

① Lemma: path lifting.

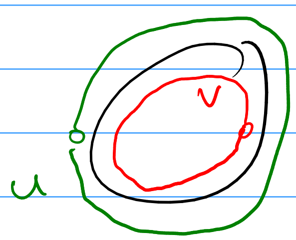
$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ I \arrow[r, "\gamma"'] \arrow[ur, dashed, "\tilde{\gamma}"] & S^1 \end{tikzcd}$$



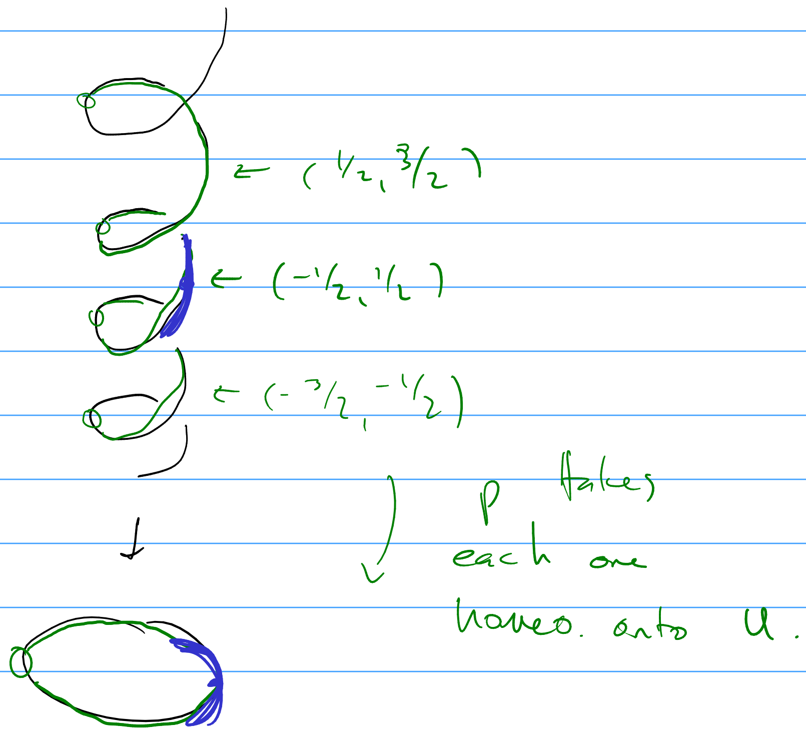
Let $\gamma: I \to S^1$ with $\gamma(0) = 1$

Then $\exists! \; \tilde{\gamma}: I \to \mathbb{R}$ with $\tilde{\gamma}(0) = 0$
and $p \circ \tilde{\gamma} = \gamma$

Pf let $U = S^1 \setminus \{-1\}$ $V = S^1 \setminus \{1\}$



If $\gamma(I)$ were contained in U,
conclusion would be easy:



also if $\gamma(I) \subset V$, although that would contradict $\gamma(0) = 1$ ...

cut I into k pieces $k >> 0$