Last worksheet: S^∞ is contractible.

Another model of S^∞:

$V := L^1([0,1])$

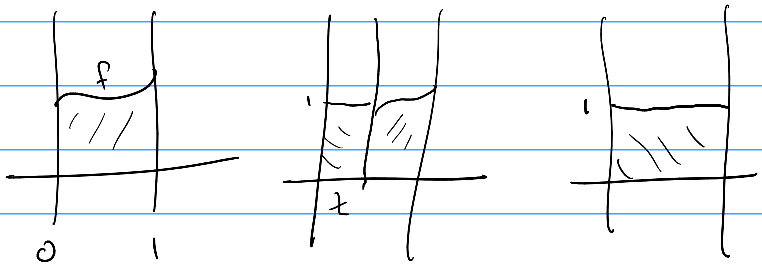
has a norm: $\|f\|$ $\int_0^1 f$

$S^\infty := \{ f \in V \mid \|f\| = 1 \}$ (not the same as last time, but just as good.)

contractible:

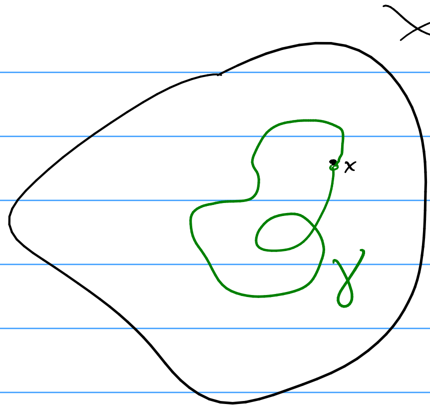
$F: S^\infty \times I \longrightarrow S^\infty$

htpy from id to const map



F(f, 0) F(f, t) F(f, 1)

The Fundamental Group

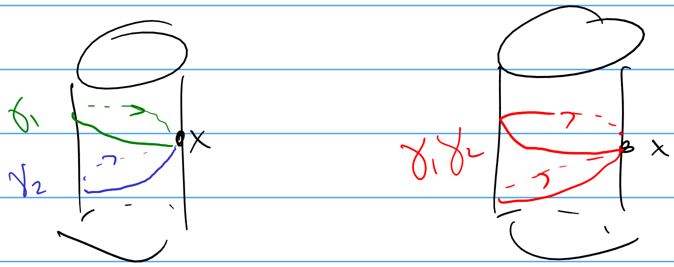


Fix a base point $x \in X$

A loop based at x

is a map $\gamma : I \longrightarrow X$
with $\gamma(0) = \gamma(1) = x$

Want to put a group str on set of loops

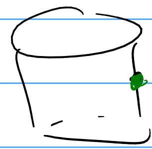


composition:
Given γ_1, γ_2, define

$$\gamma_1 \cdot \gamma_2 : I \longrightarrow X$$

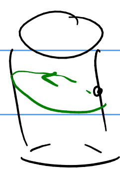
$$s \longmapsto \begin{cases} \gamma_1(2s) & 0 \leq s \leq 1/2 \\ \gamma_2(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

identity: constant loop $e(s) = x \quad \forall s \in I$



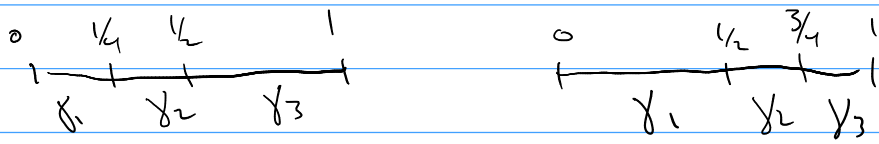
inverse: same loop backwards

$$\gamma^{-1}(s) := \gamma(1-s)$$

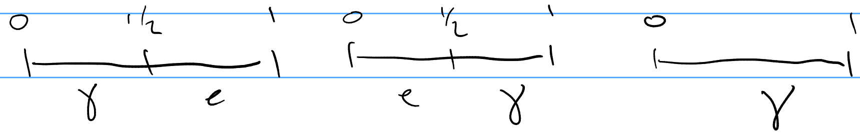


Problem: group axioms fail.

$$(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 \neq \gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$$



$$\gamma \cdot e \neq e \cdot \gamma \neq \gamma$$



$$\gamma \cdot \gamma^{-1} \neq \gamma^{-1} \cdot \gamma \neq e$$

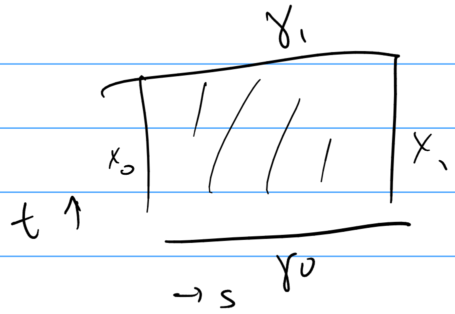
Solution: identify homotopic loops.

Def Let $\gamma_0, \gamma_1 : I \to X$ be two paths starting at $x_0 \in X$ and ending at $x_1 \in X$.

A <u>homotopy rel. endpoints</u> from γ_0 to γ_1 is a htpy

$F : I \times I \to X$ from γ_0 to γ_1

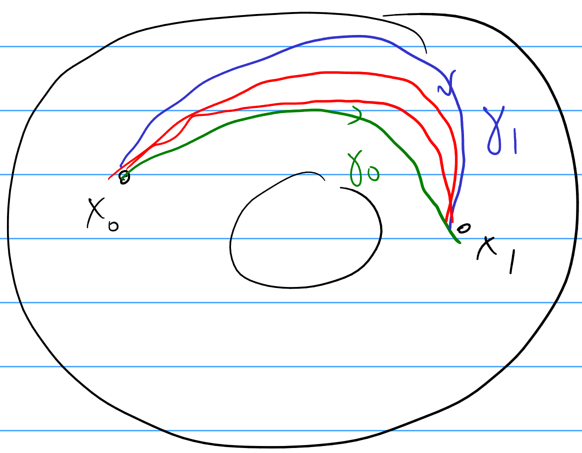
with $F(0,t) = x_0 \quad \forall t \in I$
$F(1,t) = x_1$

Picture: 

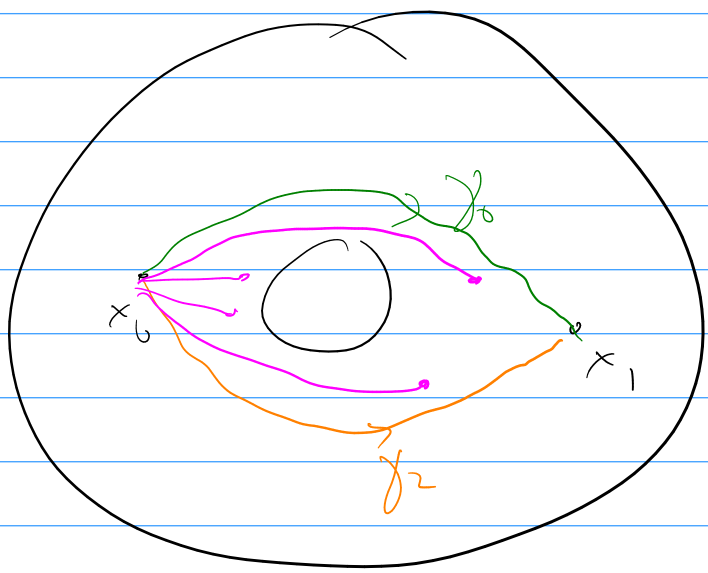
equiv: $\forall\, t \in I$, $\gamma_t := F(-, t)$

is also a path from x_0 to x_1

Examples: $X =$ annulus

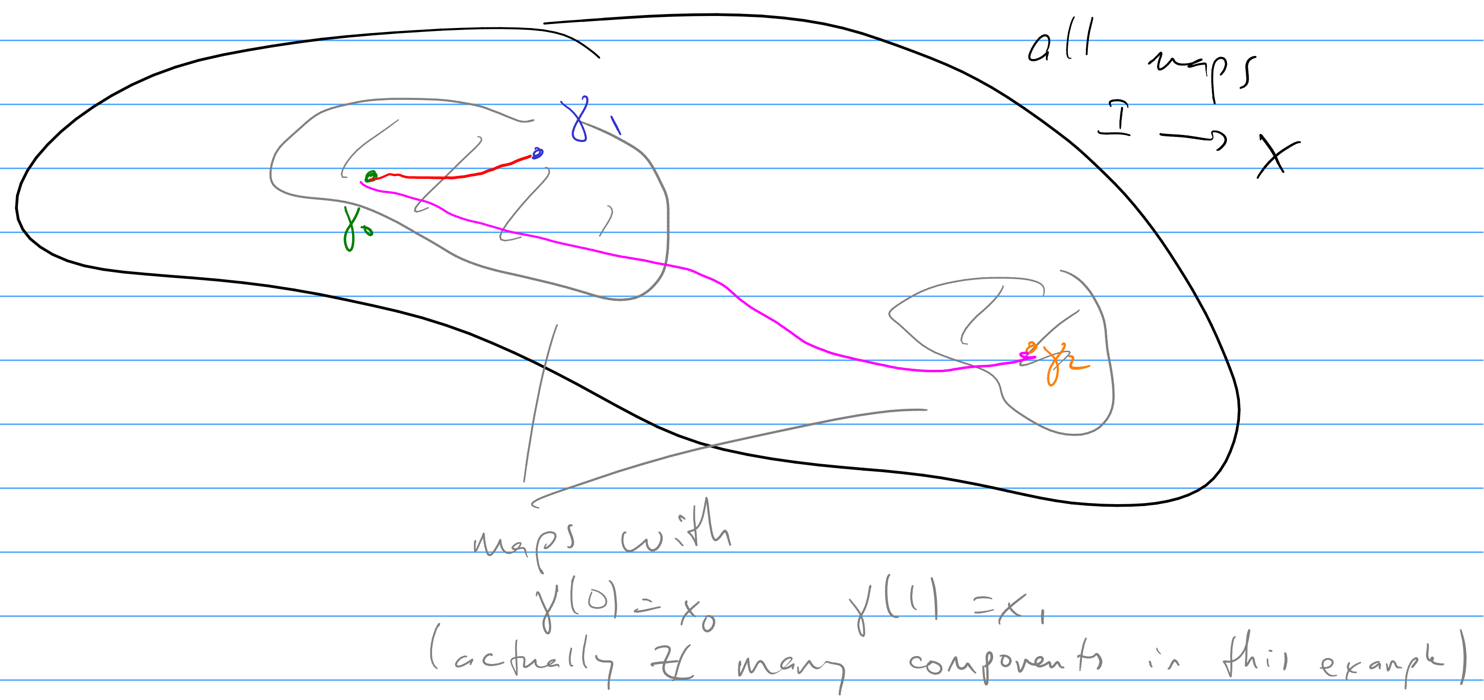


Non-example:



γ_0 and γ_2 are homotopic, but not rel. endpoints.

If we think of a htpy as a path of maps,



Def: The fundamental group (or Poincaré group)

$$\pi_1(X, x) = \{\text{loops based at } x\} \Big/ \begin{array}{l}\text{htpy} \\ \text{rel endpoints.}\end{array}$$

make it a group via

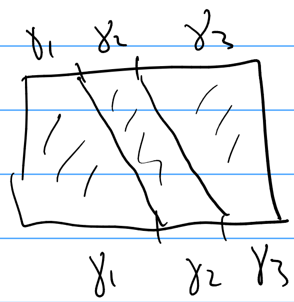
$$[\gamma_1] \cdot [\gamma_2] = [\gamma_1 \cdot \gamma_2]$$

$$\text{identity} = [e]$$

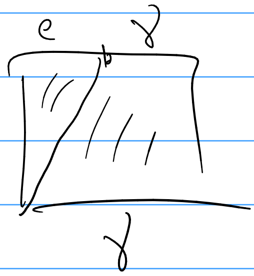
$$[\gamma]^{-1} = [\gamma^{-1}]$$

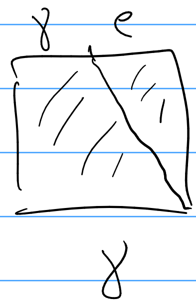
This satisfies the group axioms. Sketch:

assoc: $(\gamma_1 \cdot \gamma_2) \cdot \gamma_3 \simeq \gamma_1 \cdot (\gamma_2 \cdot \gamma_3)$



id: $e \cdot \gamma \simeq \gamma \simeq \gamma \cdot e$





inverses: $\gamma \cdot \gamma^{-1} \simeq e \simeq \gamma^{-1} \cdot \gamma$