Homotopy Equivalence

Recall: $f: X \to Y$ is a <u>homeomorphism</u>

if $\exists\, g: Y \to X$ s.t. $g \circ f = 1_X$ and $f \circ g = 1_Y$

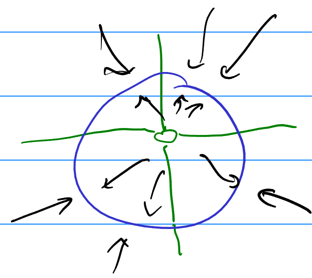
<u>Def</u> f is a <u>homotopy equivalence</u>

if $\exists\, g$ s.t. $g \circ f \simeq 1$ and $f \circ g \simeq 1$.

If $\exists$ a homotopy equivalence then we say X and Y are 'homotopy equivalent', or 'have the same homotopy type'.

We write $X \simeq Y$.

<u>Examples</u> ① the inclusion $f: S^n \hookrightarrow \mathbb{R}^{n+1} \setminus 0$ is a homotopy equivalence.



the map $g: \mathbb{R}^{n+1} \setminus 0 \to S^n$

$$x \longmapsto x/|x|$$

is a homotopy inverse to f.

check: $g \circ f = 1$ on S^n

$$f \circ g : \mathbb{R}^{n+1} \setminus 0 \longrightarrow \mathbb{R}^{n+1} \setminus 0$$

$$x \longmapsto x/|x|$$

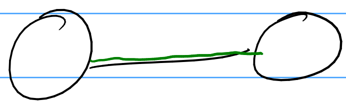
need a htpy $f \circ g \simeq 1$

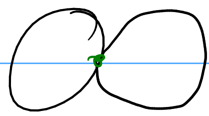
could take the straight-line htpy

$$F(x,t) = (1-t)\frac{x}{|x|} + tx$$

or we could take

$$F(x,t) = \frac{x}{|x|^t} \ \ldots$$

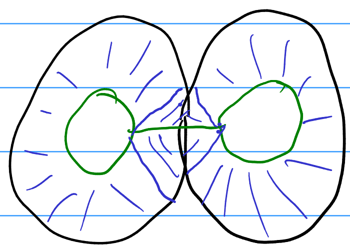
② let $X =$ 

$Y =$ 

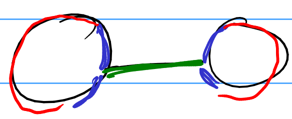
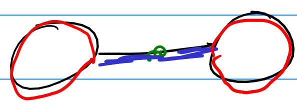
these are htpy equiv.

$f: X \to Y$ contracts the bar in the middle.

$g: Y \to X$

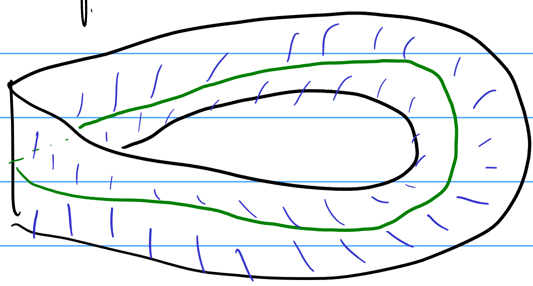


now $g \circ f \neq 1$ $\quad f \circ g \neq 1$

 $\xrightarrow{g\circ f}$ 

but they are $\simeq 1$.

(3) Möbius strip $\simeq$ circle



Def a space X is <u>contractible</u> if it's htpy equiv. to a point.

Examples: (1) $\mathbb{R}^n$ is contractible

the inclusion $0 \xhookrightarrow{f} \mathbb{R}^n$

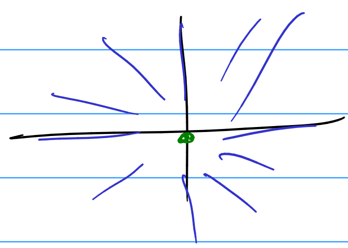
is a htpy equiv
with htpy inverse given by

$$\mathbb{R}^n \xrightarrow{g} 0$$

$$g \circ f = 1$$

$f \circ g : \mathbb{R}^n \to \mathbb{R}^n$ is the zero map

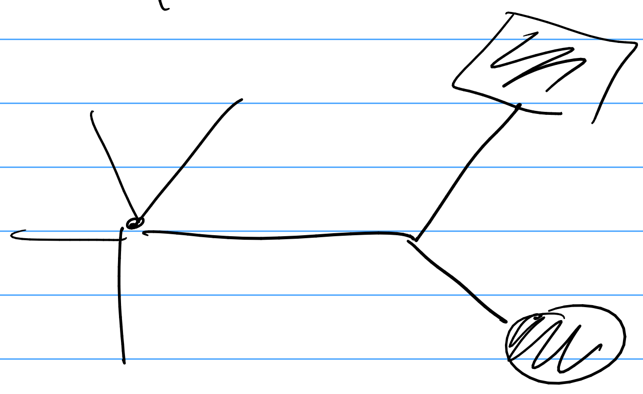
$F(x, t) = tx$ is htpy $f \circ g \simeq 1$



(2) D^n is contractible.

same proof.

(3)



is contractible

<u>Prop</u> X is contractible iff $1: X \to X$ is nullhomotopic

<u>Pf</u> worksheet.

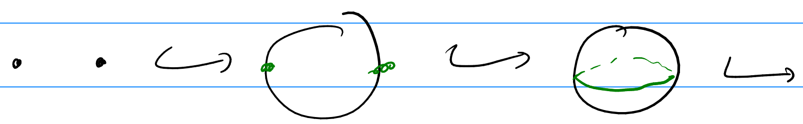
Eventually: S^n is not contractible.

Worksheet: S^∞ is contractible.

$$\mathbb{R}^\infty := \{ (x_0, x_1, x_2, \dots) : x_i \in \mathbb{R}, \text{ eventually zero.} \}$$

$$S^\infty := \{ \vec{x} \in \mathbb{R}^\infty \mid |\vec{x}| = 1 \}$$

$$S^0 \hookrightarrow S^1 \hookrightarrow S^2 \hookrightarrow S^3 \hookrightarrow \dots$$



S^∞ is the direct limit of this system.