Worksheet 9

Math 607, Connections and Characteristic Classes

Wednesday, February 24, 2021

Let E be a real or complex vector bundle on a manifold X, and let $s \in \Gamma(E)$ be a section. We have seen that there is no well-defined derivative $ds\colon TX \to E$, which is why we introduced connections. But on the set $Z \subset X$ where S vanishes, we will see that there *is* a well-defined derivative

$$ds\colon TX|_Z \to E|_Z.$$

Note that Z need not be a manifold in general.

1. Let $z \in Z$, choose a neighborhood U and sections $e_1, \ldots, e_r \in \Gamma(E|_U)$ that give a trivialization of $E|_U$, and write $s = \sum s_i e_i$, where the s_i are functions on U. We'll take the derivative of s component-wise: for a tangent vector $v \in T_z X$, define

   $$ds(v) = \sum ds_i(v)\, e_i(z) \in E_z.$$

   Check that this is well-defined: if $e'_1, \ldots, e'_r \in \Gamma(E|_U)$ is another trivialization of $E|_U$, and we write $s = \sum s'_i e'_i$, then you should verify that

   $$\sum ds_i(v)\, e_i(z) = \sum ds'_i(v)\, e'_i(z).$$

2. The section s is called *transverse* if the derivative $ds\colon TX|_Z \to E|_Z$ is surjective at each $z \in Z$. Then the implicit function theorem implies that Z is a manifold and that $TZ = \ker ds$, so ds gives an isomorphism $N_{E/Z} \cong E|_Z$. Discuss any questions that you have about this.

3. Convince yourselves that s is transverse in the sense above if and only if the graph of s, as a submanifold of the total space of E, is transverse to the zero section. Draw a little picture.