Worksheet 7, continued

Math 607, Connections and Characteristic Classes

Friday, February 12, 2021

Let $X \subset \mathbb{R}^3$ be the graph of a function: $z = f(x, y)$. Thus X is a surface, parametrized by

$$\phi \colon \mathbb{R}^2 \to \mathbb{R}^3 \qquad\qquad \phi(x, y) = (x, y, f(x, y))$$

Eventually we're going to assume that $f(0, 0) = 0$ and $f_x(0, 0) = f_y(0, 0) = 0$, so the surface passes through the origin and the tangent plane there is the xy-plane. For any surface in $\mathbb{R}^3$ and any point on that surface, we can change coordinates on $\mathbb{R}^3$ to put the point at the origin and make the tangent plane the xy-plane, so this is no loss of generality.

Last time we used the notation $\frac{\partial}{\partial x}$ all over the place, but for compactness let's use ∂_x this time, and instead of $\frac{\partial f}{\partial x}$ let's use f_x.

0. Draw a little picture to put yourselves in the right frame of mind.

1. Compute $\phi_*(\partial_x)$ and $\phi_*(\partial_y)$.

   Answer:

$$\phi_*(\partial_x) = \begin{pmatrix} 1 \\ 0 \\ f_x \end{pmatrix} \qquad\qquad \phi_*(\partial_y) = \begin{pmatrix} 0 \\ 1 \\ f_y \end{pmatrix}$$

2. Let g be the standard Riemannian metric in $\mathbb{R}^3$ – the good old dot product – and let $g' = \phi^* g$. Compute the matrix of g' in the basis ∂_x, ∂_y. Compute the volume form, or since we're on a surface you could just call it the area form:

$$dA = \sqrt{\det g'}\, dx \wedge dy.$$

Answer:

$$\begin{aligned} g' &= \begin{pmatrix} 1+f_x^2 & f_x f_y \\ f_x f_y & 1+f_y^2 \end{pmatrix} \\ \det g' &= 1 + f_x^2 + f_y^2 \\ dA &= \sqrt{1+f_x^2+f_y^2}\, dx \wedge dy \end{aligned}$$

3. View $\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$ as a section of $T\mathbb{R}^3|_X$. Find its orthogonal projection onto TX, expressed in terms of ∂_x and ∂_y.

Answer: You could use the beloved formula for orthogonal projection: if $V = (0,0,1)$ and N is the normal vector $(-f_x, -f_y, 1)$, then we want

$$V - \tfrac{V\cdot N}{N\cdot N} N = \begin{pmatrix}0\\0\\1\end{pmatrix} - \tfrac{1}{1+f_x^2+f_y^2}\begin{pmatrix}-f_x\\-f_y\\1\end{pmatrix} = \tfrac{1}{1+f_x^2+f_y^2}\begin{pmatrix}-f_x\\-f_y\\f_x^2+f_y^2\end{pmatrix}.$$

You can eyeball that as

$$W := \tfrac{1}{1+f_x^2+f_y^2}(f_x\partial_x + f_y\partial_y).$$

Or you could say that if W is the projection of V that we seek, then it's the unique linear combination of ∂_x and ∂_y satisfying

$$\begin{aligned} g(W,\partial_x) &= g(V,\partial_x) = f_x \\ g(W,\partial_y) &= g(V,\partial_y) = f_y; \end{aligned}$$

so in matrix form we want

$$\begin{aligned} W &= \begin{pmatrix} 1+f_x^2 & f_x f_y \\ f_x f_y & 1+f_y^2 \end{pmatrix}^{-1} \begin{pmatrix} f_x \\ f_y \end{pmatrix} \\ &= \tfrac{1}{1+f_x^2+f_y^2}\begin{pmatrix} 1+f_y^2 & -f_x f_y \\ -f_x f_y & 1+f_x^2 \end{pmatrix}\begin{pmatrix} f_x \\ f_y \end{pmatrix} \\ &= \tfrac{1}{1+f_x^2+f_y^2}\begin{pmatrix} f_x \\ f_y \end{pmatrix}, \end{aligned}$$

which agrees with what we found by the other method.

4. As before, we'll take the standard connection ∇ on $T\mathbb{R}^3|_X$, and hit it with the orthogonal projection $T\mathbb{R}^3|_X \to TX$ to get the Levi-Civita connection ∇' on X. Compute

$$\nabla'_x \partial_x \qquad \nabla'_x \partial_y \qquad \nabla'_y \partial_x \qquad \nabla'_y \partial_y,$$

where ∇'_x is short for ∇'_{∂_x}.

Tip: It will be convenient to let $W \in \Gamma(TX)$ be the vector field that we found in problem 3 as the projection of $\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)$, and write your answers in terms of W rather than ∂_x and ∂_y.

Answer:

$$\begin{aligned} \nabla'_x \partial_x &= f_{xx} W & \qquad \nabla'_x \partial_y &= f_{xy} W \\ \nabla'_y \partial_x &= f_{xy} W & \qquad \nabla'_y \partial_y &= f_{yy} W. \end{aligned}$$

$4\frac{1}{2}$. Continue to let $W \in \Gamma(TX)$ be the vector field that we found in problem 3. Compute $\nabla'_x W$ and $\nabla'_y W$ at the point $(0,0)$, assuming that $f_x(0,0) = f_y(0,0) = 0$.

Answer: We have

$$W = \tfrac{1}{1+f_x^2+f_y^2}(f_x \partial_x + f_y \partial_y),$$

so

$$\begin{aligned} \nabla'_x W = &-\tfrac{2f_x f_{xx} + 2 f_y f_{xy}}{(1+f_x^2+f_y^2)^2}(f_x \partial_x + f_y \partial_y) \\ &+ \tfrac{1}{1+f_x^2+f_y^2}(f_{xx}\partial_x + f_x f_{xx} W + f_{xy}\partial_y + f_y f_{xy} W) \end{aligned}$$

When we plug in $(0,0)$ and all the first derivatives drop out, this simplifies to

$$\nabla'_x W|_{(0,0)} = f_{xx}\partial_x + f_{xy}\partial_y,$$

and similarly

$$\nabla'_y W|_{(0,0)} = f_{xy}\partial_x + f_{yy}\partial_y.$$

5. Compute

$$\nabla'_x \nabla'_y \partial_x - \nabla'_y \nabla'_x \partial_x \qquad \nabla'_x \nabla'_y \partial_y - \nabla'_y \nabla'_x \partial_y$$

at the point $(0,0)$.

Answer:

$$\begin{aligned}\nabla'_x\nabla'_y\partial_x - \nabla'_y\nabla'_x\partial_x &= \nabla'_x(f_{xy}W) - \nabla'_y(f_{xx}W)\\ &= f_{xxy}W + f_{xy}\nabla'_x W - f_{xxy}W - f_{xx}\nabla'_y W.\end{aligned}$$

Evaluating at zero, this becomes

$$\begin{aligned}&0 + f_{xy}(f_{xx}\partial_x + f_{xy}\partial y) - 0 - f_{xx}(f_{xy}\partial_x + f_{yy}\partial_y)\\ &= (f_{xy}^2 - f_{xx}f_{yy})\partial_y.\end{aligned}$$

Similarly, we find that

$$(\nabla'_x\nabla'_y\partial_y - \nabla'_y\nabla'_x\partial_y)|_{(0,0)} = (f_{yy}f_{xx} - f_{xy}^2)\partial_x.$$

So we're happy to see that the curvature matrix is skew-symmetric:

$$\begin{pmatrix} 0 & f_{yy}f_{xx} - f_{xy}^2 \\ f_{xy}^2 - f_{xx}f_{yy} & 0 \end{pmatrix} dx \wedge dy.$$

And also at $(0,0)$ we have $dA = dx \wedge dy$ so everything is copacetic.

6. Suppose we have the graph of a function of one variable, $y = h(x)$, and that $h(0) = h'(0) = 0$. Convince yourselves that the radius of the osculating circle at the origin is $|1/h''(0)|$. Draw a little picture.

7. If $(a, b, 0) \in \mathbb{R}^3$ is a unit tangent vector to our surface X at the origin, we define the (signed) curvature in that direction to be 1 over the (signed) radius of the osculating circle of the curve obtained by slicing X with the plane spanned by $(a, b, 0)$ and the normal vector $(0, 0, 1)$. Convince yourselves that it equals

$$a^2 f_{xx}(0,0) + 2ab f_{xy}(0,0) + b^2 f_{yy}(0,0).$$

8. That was the quadratic form associated to the symmetric matrix

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.$$

Convince yourselves that if the signed curvature is maximized in one direction and minimized in another, then the product of those two curvatures is the determinant of the matrix:

$$f_{xx}f_{yy} - f_{xy}^2.$$

This agrees with the Pfaffian of the curvature matrix that we found earlier, up to a sign.