

Worksheet 7, continued

Math 607, Connections and Characteristic Classes

Monday, February 8, 2021

Let $X \subset \mathbb{R}^3$ be the graph of a function: $z = f(x, y)$. Thus X is a surface, parametrized by

$$\phi: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \qquad \phi(x, y) = (x, y, f(x, y))$$

Eventually we're going to assume that $f(0, 0) = 0$ and $f_x(0, 0) = f_y(0, 0) = 0$, so the surface passes through the origin and the tangent plane there is the xy -plane. For any surface in $\mathbb{R}^3$ and any point on that surface, we can change coordinates on $\mathbb{R}^3$ to put the point at the origin and make the tangent plane the xy -plane, so this is no loss of generality.

Last time we used the notation $\frac{\partial}{\partial x}$ all over the place, but for compactness let's use ∂_x this time, and instead of $\frac{\partial f}{\partial x}$ let's use f_x .

0. Draw a little picture to put yourselves in the right frame of mind.
1. Compute $\phi_*(\partial_x)$ and $\phi_*(\partial_y)$.

Answer:

$$\phi_*(\partial_x) = \begin{pmatrix} 1 \\ 0 \\ f_x \end{pmatrix} \qquad \phi_*(\partial_y) = \begin{pmatrix} 0 \\ 1 \\ f_y \end{pmatrix}$$

2. Let g be the standard Riemannian metric in $\mathbb{R}^3$ – the good old dot product – and let $g' = \phi^*g$. Compute the matrix of g' in the basis ∂_x, ∂_y . Compute the volume form, or since we're on a surface you could just call it the area form:

$$dA = \sqrt{\det g'} dx \wedge dy.$$

Answer:

$$\begin{aligned} g' &= \begin{pmatrix} 1 + f_x^2 & f_x f_y \\ f_x f_y & 1 + f_y^2 \end{pmatrix} \\ \det g' &= 1 + f_x^2 + f_y^2 \\ dA &= \sqrt{1 + f_x^2 + f_y^2} dx \wedge dy \end{aligned}$$

3. View $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ as a section of $T\mathbb{R}^3|_X$. Find its orthogonal projection onto TX , expressed in terms of ∂_x and ∂_y .

Answer: You could use the beloved formula for orthogonal projection: if $V = (0, 0, 1)$ and N is the normal vector $(-f_x, -f_y, 1)$, then we want

$$V - \frac{V \cdot N}{N \cdot N} N = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{1 + f_x^2 + f_y^2} \begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix} = \frac{1}{1 + f_x^2 + f_y^2} \begin{pmatrix} -f_x \\ -f_y \\ f_x^2 + f_y^2 \end{pmatrix}.$$

You can eyeball that as

$$W := \frac{1}{1 + f_x^2 + f_y^2} (f_x \partial_x + f_y \partial_y).$$

Or you could say that if W is the projection of V that we seek, then it's the unique linear combination of ∂_x and ∂_y satisfying

$$\begin{aligned} g(W, \partial_x) &= g(V, \partial_x) = f_x \\ g(W, \partial_y) &= g(V, \partial_y) = f_y; \end{aligned}$$

so in matrix form we want

$$\begin{aligned} W &= \begin{pmatrix} 1 + f_x^2 & f_x f_y \\ f_x f_y & 1 + f_y^2 \end{pmatrix}^{-1} \begin{pmatrix} f_x \\ f_y \end{pmatrix} \\ &= \frac{1}{1 + f_x^2 + f_y^2} \begin{pmatrix} 1 + f_y^2 & -f_x f_y \\ -f_x f_y & 1 + f_x^2 \end{pmatrix} \begin{pmatrix} f_x \\ f_y \end{pmatrix} \\ &= \frac{1}{1 + f_x^2 + f_y^2} \begin{pmatrix} f_x \\ f_y \end{pmatrix}, \end{aligned}$$

which agrees with what we found by the other method.

4. As before, we'll take the standard connection ∇ on $T\mathbb{R}^3|_X$, and hit it with the orthogonal projection $T\mathbb{R}^3|_X \rightarrow TX$ to get the Levi-Civita connection ∇' on X . Compute

$$\nabla'_x \partial_x \quad \nabla'_x \partial_y \quad \nabla'_y \partial_x \quad \nabla'_y \partial_y,$$

where ∇'_x is short for ∇'_{∂_x} .

Tip: It will be convenient to let $W \in \Gamma(TX)$ be the vector field that we found in problem 3 as the projection of $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and write your answers in terms of W rather than ∂_x and ∂_y .

- 4 $\frac{1}{2}$. Continue to let $W \in \Gamma(TX)$ be the vector field that we found in problem 3. Compute $\nabla'_x W$ and $\nabla'_y W$ at the point $(0,0)$, assuming that $f_x(0,0) = f_y(0,0) = 0$.

5. Compute

$$\nabla'_x \nabla'_y \partial_x - \nabla'_y \nabla'_x \partial_x \quad \nabla'_x \nabla'_y \partial_y - \nabla'_y \nabla'_x \partial_y$$

at the point $(0,0)$.

6. Suppose we have the graph of a function of one variable, $y = h(x)$, and that $h(0) = h'(0) = 0$. Convince yourselves that the radius of the osculating circle at the origin is $|1/h''(0)|$. Draw a little picture.
7. If $(a, b, 0) \in \mathbb{R}^3$ is a unit tangent vector to our surface X at the origin, we define the (signed) curvature in that direction to be 1 over the (signed) radius of the osculating circle of the curve obtained by slicing X with the plane spanned by $(a, b, 0)$ and the normal vector $(0, 0, 1)$. Convince yourselves that it equals

$$a^2 f_{xx}(0,0) + 2ab f_{xy}(0,0) + b^2 f_{yy}(0,0).$$

8. That was the quadratic form associated to the symmetric matrix

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.$$

Convince yourselves that if the signed curvature is maximized in one direction and minimized in another, then the product of those two curvatures is the determinant of the matrix:

$$f_{xx}f_{yy} - f_{xy}^2.$$

This should be the the Pfaffian of the curvature matrix that you found earlier.