# Worksheet 4

Math 607, Connections and Characteristic Classes

Friday, January 16, 2021

If this is too much for one day we can come back to it next time, because it's beautiful stuff.

Let $V = \mathbb{R}^{2n}$, and let $J$ be the $2n \times 2n$ block matrix

$$J = \left(\begin{array}{c|c} 0 & -1 \\ \hline 1 & 0 \end{array}\right).$$

1. If we view $J$ as a map $V \to V$, then it's a complex structure: $J^2 = -1$. Convince yourselves that

   $$\mathrm{GL}_n(\mathbb{C}) = \{A \in \mathrm{GL}_{2n}(\mathbb{R}) : AJ = JA\}.$$

   Then argue that $AJ = JA$ implies $\det A > 0$, so a complex structure determines an orientation.

   Hint: For the second part, the way I know is to show that $A$ must be of the form $\left(\begin{smallmatrix} B & -C \\ C & B \end{smallmatrix}\right)$, then conjugate by $\left(\begin{smallmatrix} 1 & i \\ 1 & -i \end{smallmatrix}\right)$, then take determinants. But maybe you know a better way, or can think of one?

2. If we view $J$ as an element of $\Lambda^2 V^*$, then it's a symplectic form: $J \wedge \cdots \wedge J \neq 0$. Convince yourselves that the group of automorphisms of $V$ that preserve the symplectic form $J$ – called the symplectic group – is

   $$\mathrm{Sp}_{2n}(\mathbb{R}) = \{A \in \mathrm{GL}_{2n}(\mathbb{R}) : A^\top J A = J\}.$$

   Then argue that $\mathrm{Sp}_{2n}(\mathbb{R}) \subset \mathrm{SL}_{2n}(\mathbb{R})$, so a symplectic form determines a volume form.

3. We also have the good old orthogonal group

   $$\mathrm{O}(2n) = \{A \in \mathrm{GL}_{2n}(\mathbb{R}) : A^\top A = 1\}.$$

   Show that $\mathrm{O}(2n) \cap \mathrm{GL}_n^+(\mathbb{R})$ is contained in $\mathrm{SL}_{2n}(\mathbb{R})$, so a Riemannian metric and an orientation determine a volume form. (This is the special orthogonal group $\mathrm{SO}(2n)$.)

4. Show that
$$\mathrm{O}(2n) \cap \mathrm{GL}_n(\mathbb{C}) = \mathrm{Sp}_{2n}(\mathbb{R}),$$
so if a vector bundle $E$ admits a Riemannian metric and a compatible complex structure, then they determine a symplectic form. Similarly,
$$\mathrm{Sp}_{2n}(\mathbb{R}) \cap \mathrm{GL}_n(\mathbb{C}) = \mathrm{O}(2n) \qquad \mathrm{O}(2n) \cap \mathrm{Sp}_{2n}(\mathbb{R}) = \mathrm{GL}_n(\mathbb{C}),$$
so any two structures determine the third.