Worksheet 13

Math 607, Connections and Characteristic Classes

Monday, March 8, 2021

To prepare for Wednesday's lecture, we want to define Čech cohomology of X with coefficients in a group G. Let $\mathcal{U} = \{U_i\}$ be an open cover of X. First we define cochains:

- An element $\varphi \in \check{C}^0(\mathcal{U}, G)$ is a set of maps $\varphi_i \colon U_i \to G$ for each $U_i \in \mathcal{U}$.
- An element $\psi \in \check{C}^1(\mathcal{U}, G)$ is a set of maps $\psi_{ij} \colon U_i \cap U_j \to G$ for each $U_i, U_j \in \mathcal{U}$.
- An element $\xi \in \check{C}^2(\mathcal{U}, G)$ is a set of maps $\xi_{ijk} \colon U_i \cap U_j \cap U_k \to G$ for each $U_i, U_j, U_k \in \mathcal{U}$.
- We can go on if we want.

These are groups under pointwise multiplication. We can take several flavors of maps here: either locally constant maps, or continuous maps if G is a topological group, or smooth maps if X is a manifold and G is a Lie group, or holomorphic maps if X is a complex manifold and G is a complex Lie group.

Next we define coboundaries:

- $d_0 \colon \check{C}^0(\mathcal{U}, G) \to \check{C}^1(\mathcal{U}, G)$ is given by $(d\varphi)_{ij} = \varphi_j \cdot \varphi_i^{-1}$, which is defined on $U_i \cap U_j$. Here $\cdot$ means pointwise multiplication of maps $U_i \cap U_j \to G$.
- $d_1 \colon \check{C}^1(\mathcal{U}, G) \to \check{C}^2(\mathcal{U}, G)$ is given by $(d\psi)_{ijk} = \psi_{ij} \cdot \psi_{ik}^{-1} \cdot \psi_{jk}$, which is defined on $U_i \cap U_j \cap U_k$.
- $d_2 \colon \check{C}^2(\mathcal{U}, G) \to \check{C}^3(\mathcal{U}, G)$ is given by $(d\xi)_{ijkl} = \xi_{ijk} \cdot \xi_{ijl}^{-1} \cdot \xi_{ikl} \cdot \xi_{jkl}^{-1}$, which is defined on $U_i \cap U_j \cap U_k \cap U_l$.

(Questions on the next page.)

1. Convince yourselves that if G is Abelian then the d_is are group homomorphisms, and $d^2 = 0$.

   We define $\check{H}^i(\mathcal{U}, G) = \ker d_i / \operatorname{im} d_{i-1}$, and $\check{H}^i(X, G)$ as the direct limit of $\check{H}^i(\mathcal{U}, G)$ over all open covers $\mathcal{U}$, partially ordered by refinement.

2. Convince yourselves that if G is not Abelian then the d_is are not homomorphisms, and that $d_1 \circ d_0 = 1$ but $d_2 \circ d_1 \neq 1$.

3. In any case, convince yourselves that $\check{H}^0(\mathcal{U}, G) := \ker d_0$ is the same as maps from X to G. Observe that this is a group.

4. If G is not Abelian then neither the 1-cocycles ($\ker d_1$) nor the 1-coboundaries ($\operatorname{im} d_0$) form a group, but the group of 0-cochains does act on the set of 1-cocyles:

   $$(\varphi \cdot \psi)_{ij} = \varphi_j \cdot \psi_{ij} \cdot \varphi_i^{-1}.$$

   We define $\check{H}^1(\mathcal{U}, G)$ to be the quotient of this set by this action. This is not a group, but it is a pointed set, with the distinguished element represented by the trivial cocycle $\psi_{ij} \equiv 1$.

   Convince yourselves that if G is Abelian then this agrees with our previous definition of $\check{H}^1(\mathcal{U}, G)$.

5. Challenge: Show that if X admits partitions of unity and we take continuous maps with values in $\mathbb{R}$, or smooth maps if X is a manifold, then $\check{H}^i(X, \mathbb{R}) = 0$ for $i > 0$.