Note on Transversality

Math 607, Connections and Characteristic Classes

Friday, March 5, 2021

Suppose we have smooth manifolds and smooth maps as shown:

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

Given points $x \in X$ and $y \in Y$ with $f(x) = g(y) =: z$, we have the derivatives

$$Df\colon T_x X \to T_z Z$$
$$Dg\colon T_y Y \to T_z Z.$$

We say that f and g are *transverse* if for all such x and y, the images of Df and Dg span $T_z Z$:

$$Df(T_x X) + Dg(T_y Y) = T_z Z.$$

If f and g are embeddings then this agrees with the usual notion of two submanifolds being transverse, and if f is arbitrary and g is an embedding then this agrees with the usual notion of a map being transverse to a submanifold.

If f and g are transverse then the fiber product $X \times_Z Y$ is a smooth manifold. It has the expected dimension, namely $\dim X + \dim Y - \dim Z$, and the tangent space at a point $T_{(x,y)}(X \times_Z Y)$ is the kernel of the surjection

$$T_x X \oplus T_y Y \xrightarrow{(Df\ \ Dg)} T_z Z.$$

You can check that the following are equivalent:

1. f and g are transverse.
2. The map $f \times g\colon X \times Y \to Z \times Z$ is transverse to the diagonal $\Delta \subset Z \times Z$. (And $X \times_Z Y = (f \times g)^{-1}(\Delta)$.)
3. The map $f \times 1\colon X \times Y \to Z \times Y$ is transverse to the graph $\Gamma_g \subset Z \times Y$. (And $X \times_Z Y = (f \times 1)^{-1}(\Gamma_g)$.)
4. The submanifolds $\Gamma_f \times Y$ and $X \times \Gamma_g$ in $X \times Z \times Y$ are transverse. (And $X \times_Y Z$ is their intersection.)