Worksheet 3

Math 607, Homological Algebra

Friday, April 3, 2020

Soon we'll see that a finitely generated module over a Noetherian commutative ring is projective iff it's flat iff it's locally free. In this worksheet you'll get a feeling for some relevant examples.

0. Let R be an integral domain and $I \subset R$ an ideal. Convince yourself that I is a free module if and only if it's a (non-zero) principal ideal.

1. Let $R = \mathbb{Z}[\sqrt{-5}]$, and let $I = (2, 1+\sqrt{-5})$. Show that I is a locally free module but not a free module, as follows:

   Locally free. Recall that $2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$ in R. Show that I becomes principal when you tensor up to $R[1/2]$ or $R[1/3]$. For every maximal ideal $\mathfrak{m}$, either $2 \notin \mathfrak{m}$ or $3 \notin \mathfrak{m}$, so $R_{\mathfrak{m}}$ is a localization of $R[1/2]$ or $R[1/3]$.

   Not free, i.e. not principal. We want to show that both 2 and $1+\sqrt{-5}$ are irreducible, so if some $\alpha \in R$ divides both of them then it's a unit. Consider the norm $N(x+y\sqrt{-5}) = x^2+5y^2 \in \mathbb{Z}$. For $\alpha, \beta \in R$ we have $N(\alpha\beta) = N(\alpha)N(\beta)$, and α is a unit iff $N(\alpha) = 1$. We see that there's no $\alpha \in R$ with $N(\alpha) = 2$ or 3.

1′. Imagine playing the same game with

$$R = k[x,y]/(y^2 = x^3 - x),$$

   where k is a field of characteristic $\neq 2$, and $I = (x, y)$. You would use the fact that $y^2 = x(x^2-1)$ in R and tensor up to $R[1/x]$ and $R[1/(x^2-1)]$. You would observe that every $f \in R$ can be written uniquely as $g(x) + h(x)y$, use the norm $N(g+hy) = g^2 + h^2(x^3-x) \in k[x]$, and argue that there's no f with $N(f) = x$ or $x+1$ or $x-1$, so x and y are irreducible.

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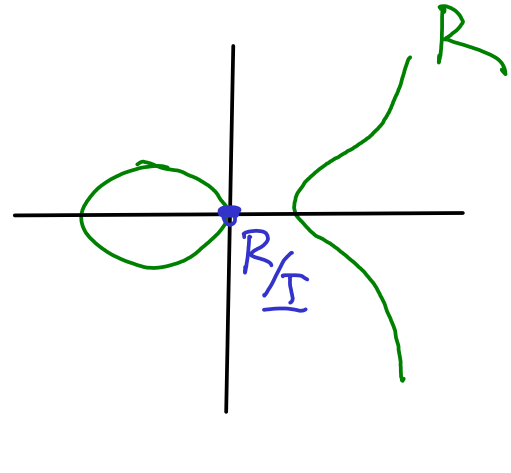

2. We've seen that $k[x, 1/x]$ is a flat $k[x]$-module. Understand why it's not finitely generated. Show that it's not projective.

   Hint: Consider the surjection $p\colon k[x, y] \twoheadrightarrow k[x, y]/(xy - 1) \cong k[x, 1/x]$. If the target were projective then it would have a section s, but show that it doesn't. Think carefully, because p is a ring homomorphism but s only has to be a $k[x]$-module homomorphism.

2′. We've seen that $\mathbb{Z}[1/2]$ is a flat $\mathbb{Z}$-module. Understand why it's not finitely generated. Show that it's not projective.

   Hint: Consider the surjection $p\colon \mathbb{Z}[y] \twoheadrightarrow \mathbb{Z}[y]/(2y - 1) \cong \mathbb{Z}[1/2]$. If the target were projective then it would have a section, but show that it doesn't. Think carefully, because p is a ring homomorphism but s only has to be a $\mathbb{Z}$-module homomorphism.