# Worksheet 25

Math 607, Homological Algebra

Friday, June 5, 2020

Let $R = k[x_1, \ldots, x_n]$, let $\mathfrak{m} = (x_1, \ldots, x_n)$, let $I = (f_1, \ldots, f_\ell) \subset \mathfrak{m}$, and let $J$ be the Jacobian matrix

$$J = \begin{pmatrix} \partial f_1/\partial x_1 & \ldots & \partial f_\ell/\partial x_1 \\ \vdots & \ddots & \vdots \\ \partial f_1/\partial x_n & \ldots & \partial f_\ell/\partial x_n \end{pmatrix}.$$

Consider the diagram

$$\begin{tikzcd} R^\ell \arrow[r, "J"] \arrow[d, two heads, "( f_1 \; \ldots \; f_\ell )"'] & R^n \arrow[d, two heads, "( x_1 \; \ldots \; x_n )"] \\ I \arrow[r, hook] & \mathfrak{m} \end{tikzcd}$$

which doesn't quite commute: if

$$f_j = (\text{linear term}) + (\text{quadratic term}) + (\text{cubic term}) + \cdots,$$

then from Euler's theorem we know that

$$\sum x_i \frac{\partial f_j}{\partial x_i} = (\text{linear term}) + 2 \cdot (\text{quadratic term}) + 3 \cdot (\text{cubic term}) + \cdots.$$

But convince yourself that if we tensor with $R/\mathfrak{m}$, then $\mathfrak{m}$ becomes $\mathfrak{m}/\mathfrak{m}^2$ and gets rid of those higher-order terms, yielding a commutative diagram

$$\begin{tikzcd} k^\ell \arrow[r, "J(0)"] \arrow[d, two heads] & k^n \arrow[d, "\cong"] \\ I/\mathfrak{m}I \arrow[r] & \mathfrak{m}/\mathfrak{m}^2. \end{tikzcd}$$

(If you're not used to the fact that $\mathfrak{m} \otimes R/\mathfrak{m} \cong \mathfrak{m}/\mathfrak{m}^2$, take the exact sequence

$$0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0$$

and tensor with $\mathfrak{m}$. The image of $\mathfrak{m} \otimes \mathfrak{m} \to \mathfrak{m}$ is $\mathfrak{m}^2 \subset \mathfrak{m}$.)