Worksheet 20

Math 607, Homological Algebra

Friday, May 22, 2020

Let R be a ring, let M be a right R-module, let N be a left R-module, and take projective resolutions

$$\cdots \to P^{-2} \xrightarrow{d} P^{-1} \xrightarrow{d} P^{0} \to M \to 0$$
$$\cdots \to Q^{-2} \xrightarrow{d'} Q^{-1} \xrightarrow{d'} Q^{0} \to N \to 0$$

with slightly different indexing than we've used before. Form a double complex

$$C^{p,q} = P^{p} \otimes_R Q^{q},$$

with horizontal differentials $d_{\text{horz}} := d \otimes 1$ and vertical differentials $d_{\text{vert}} := 1 \otimes d'$ which commute. Let T denote the total complex.

1. Draw a bit of the E_0 page of the spectral sequence that computes $H^*(T)$. (It lives in the third quadrant! The differentials go up!)
2. Pass to the E_1 page and see that it's just one row, which is $P^\bullet \otimes N$.
3. Pass to the E_2 page and see that the differentials have nowhere left to go, and we get $H^i(T) = \operatorname{Tor}^R_{-i}(M,N)$, where the Tor was computed by resolving M.
4. If you transposed everything, you would get $H^i(T) = \operatorname{Tor}^R_{-i}(M,N)$, where the Tor was computed by resolving N.
5. Think about what details you'd want to see (or check yourself) to be convinced that this proof is legitimate.