Worksheet 19

Math 607, Homological Algebra

Monday, May 18, 2020

1. Let $R = k[x, y]$, let $M = R/x$, let $S = R/xy$, and let $N = S/y$. Draw a picture.

2. If we regard N as an R-module then it's isomorphic to R/y. We see that $M \otimes_R N = R/(x, y)$. Using the free resolution

$$0 \to R \xrightarrow{y} R \to R/y \to 0,$$

see that $\mathrm{Tor}^R_{\geq 1}(M, N) = 0$.

3. We see that $M \otimes_R S = S/x$. Using this free resolution of S as an R-module

$$0 \to R \xrightarrow{xy} R \to S \to 0,$$

see that $\mathrm{Tor}^R_1(M, S) = S/x$ as well, and $\mathrm{Tor}^R_{\geq 2}(M, S) = 0$.

4. Using this free resolution of N as an S-module

$$\cdots \to S \xrightarrow{x} S \xrightarrow{y} S \xrightarrow{x} S \xrightarrow{y} S \to N \to 0,$$

see that

$$\mathrm{Tor}^S_i(S/x, N) = \begin{cases} k & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd,} \end{cases}$$

where k is short for $S/(x, y)$.

5. Write out the E_2 page of the change-of-rings spectral sequence

$$E_2^{p,q} = \mathrm{Tor}^S_{-p}(\mathrm{Tor}^R_{-q}(M, S), N) \quad \Longrightarrow \quad \mathrm{Tor}^R_{-p-q}(M, N).$$

It should live in the third quadrant. Conclude that all the differentials on the must be isomorphisms.