Worksheet 18

Math 607, Homological Algebra

Wednesday, May 15, 2020

1. Sketch the lines $x = 0$ and $y = 0$ in the xy-plane.

   Let $R = k[x, y]$. Convince yourself that
   $$0 \to R/y \xrightarrow{x} R/xy \to R/x \to 0$$
   is an exact sequence of R-modules, as you did on Worksheet 2.

   Compute
   $$\mathrm{Ext}^1(R/x, R/y)$$
   using the free resolution
   $$0 \to R \xrightarrow{x} R \to R/x \to 0.$$
   Conclude that the extension above is the only non-split extension of R/x by R/y, up to isomorphism.

2. Sketch the line $y = 0$ and the parabola $y = x^2 - 1$. Show that
   $$\mathrm{Ext}^1(R/(x^2 - 1 - y), R/y) \cong R/(x - 1, y) \oplus R/(x + 1, y)$$
   using the free resolution
   $$0 \to R \xrightarrow{x^2-1-y} R \to R/(x^2 - 1 - y) \to 0.$$
   It may be helpful to view R/y as $k[x]$.

   Geometrically, this Ext^1 is a copy of k at $(1, 0)$ and a copy at $(-1, 0)$, and the two numbers tell you how to take the trivial line bundle on $y = 0$ to the trivial line bundle on $y = x^2 - 1$ and glue them together at those two points. If both numbers are non-zero, you get a line bundle on the union of the two curves, a.k.a. a projective module of rank 1 over $R/y(x^2-1-y)$. If both are zero, you get the direct sum. If one is non-zero and the other is zero, the extension is projective at one point and looks like the direct sum at the other point.

   Considering non-split extensions up to isomorphism (rather than equivalence), we can see a
   $$(k^2 \setminus 0)/k^* = \mathbb{P}^1_k$$
   worth of extensions.