Worksheet 16

Math 607, Homological Algebra

Monday, May 11, 2020

1. Regard k^9 as the space of 3×3 matrices with entries in k, and let $X \subset k^9$ be the space of matrices of rank ≤ 1, that is, where all the 2×2 minors vanish. Let $I \subset k[x_1, \ldots, x_9]$ be the ideal that cuts out X, that is, the ideal generated by the 2×2 minors. Get this going in Macaulay2 as follows:

   ```
   R = QQ[x_1..x_9]
   M = genericMatrix(R,3,3)
   I = minors(2,M)
   ```

   Check that I is a radical ideal by asking the computer if $\sqrt{I}$ is equal to I:

   ```
   radical I == I
   ```

2. A 3×3 matrix has nine 2×2 minors. But could we have gotten away with fewer generators for I? Check that we really need them all by tensoring with the residue field at the origin:

   ```
   k = R/ideal(x_1..x_9)
   (module I) ** k
   ```

3. The generators of I do not form a regular sequence, because the Koszul complex has a bunch of non-vanishing homology. See how much:

   ```
   K = koszul gens I
   HH_1 K == 0
   HH_2 K == 0
   -- etc
   ```

   By the theorem we proved today, that means any maximal regular sequence in I has length 4.

4. Mess around and find regular sequence in I of length 4. For example, the first two generators form a regular sequence, but the first three don't:

   ```
   HH_1 koszul matrix{{I_0, I_1}} == 0 -- it's zero
   HH_1 koszul matrix{{I_0, I_1, I_2}} == 0 -- not zero
   ```

   Append some more generators to `I_0` and `I_1` and see if you can get to a sequence of length 4.