Worksheet 10

Math 607, Homological Algebra

Wednesday, April 22, 2020

Let R be an integral domain and let K be its field of fractions. If you want you can think $R = \mathbb{Z}$ and $K = \mathbb{Q}$, but also think about $R = k[x, y]$ and $K = k(x, y)$.

If M is an R-module, the *torsion submodule* of M is

$$T(M) = \{m \in M : rm = 0 \text{ for some non-zero } r \in R\}.$$

We say that M is torsion if $T(M) = M$, and that M is torsion-free if $T(M) = 0$.

1. Convince yourself that the kernel of the natural map

$$M = M \otimes R \to M \otimes K$$

   is exactly $T(M)$.

2. Consider the exact sequence

$$0 \to R \to K \to K/R \to 0,$$

   and apply $M \otimes -$ to get a long exact sequence of Tors. Conclude that

$$\mathrm{Tor}_1^R(M, K/R) = T(M)$$

   and

$$\mathrm{Tor}_{\geq 2}^R(M, K/R) = 0.$$

   Hint: K is a flat R-module.

3. Let

$$0 \to L \to M \to N \to 0$$

   be a short exact sequence of R-modules, and apply $- \otimes K/R$ to get a long exact sequence of Tors.

   Conclude that if L is torsion then the sequence

$$0 \to T(L) \to T(M) \to T(N) \to 0$$

   is exact. I once needed this fact in a paper and struggled to find a reference.

   If you have time, find a counterexample when L is not torsion.