Solutions to Midterm 2

1. (a) Show that if G is a group and $H \subset G$ is a subgroup of index 2 then H is normal.

   Solution: By definition, H has index 2 if and only if it has exactly two left (or right) cosets. Let $g \in G \setminus H$. Because $g \notin H$, we have $gH \cap H = \varnothing$, and by hypothesis $G = H \cup gH$. Similarly $Hg \cap H = \varnothing$ and $G = H \cup Hg$. Thus $gH = G \setminus H = Hg$, so H is normal.

   (b) Give an example of a group G and subgroup $H \subset G$ of index 3 that is *not* normal.

   Solution: Let $G = D_3$ and $H = \{1, s\}$. Then $|G| = 6$ and $|H| = 3$, so the index of H is $6/3 = 2$. But

   $$rHr^{-1} = \{rr^{-1}, rsr^{-1}\} = \{1, sr\} \neq H.$$

   (c) Give an example of a group G and subgroup $H \subset G$ of index 3 that *is* normal. Describe the quotient.

   Solution: Let $G = \mathbb{Z}/6$ and $H = \{0, 3\}$. Then $|G| = 6$ and $|H| = 3$, so the index of H is $6/3 = 2$. But G is Abelian, so every subgroup is normal. The quotient is a group of order 3, hence is isomorphic to $\mathbb{Z}/3$.

2. (a) Let G be a group, let A be an Abelian group, and let $\varphi\colon G \to A$ be a homomorphism. Show that if two elements $x, y \in G$ are conjugate then $\varphi(x) = \varphi(y)$.

   Solution: Since x and y are conjugate, we can write $y = gxg^{-1}$ for some $g \in G$. Thus

   $$\varphi(y) = \varphi(g)\varphi(x)\varphi(g)^{-1} = \varphi(x)\varphi(g)\varphi(g)^{-1} = \varphi(x),$$

   where in the first equality we have used the fact that φ is a homomorphism, and in the second we have used the fact that A is Abelian.

(b) Let G be a group, let $H \subset G$ be a *normal* subgroup of index 3, and let $g \in G$. Show that g is conjugate to g^{-1} then $g \in H$.

Hint: Observe that $G/H \cong \mathbb{Z}/3$, and apply part (a) to the natural projection $\varphi \colon G \to G/H$.

Solution: Since H is a normal subgroup of index 3, G/H is a group of order 3, so $G/H \cong \mathbb{Z}/3$. Letting $\varphi \colon G \to G/H$ be the natural projection, by part (a) we have $\varphi(g) = \varphi(g^{-1}) = \varphi(g)^{-1}$. But the only element of $\mathbb{Z}/3$ with this property is the identity, so $g \in \ker \varphi = H$.

(c) Give an example of a group G, a subgroup $H \subset G$ of index 3 that is *not* normal, and an element $g \in G$ such that g is conjugate to g^{-1} but $g \notin H$.

Solution: Let $G = D_3$, $H = \{1, s\}$, and $g = r$. Then $srs^{-1} = srs = s^2 r^{-1} = r^{-1}$, so r is conjugate to r^{-1}.

3. (a) Show that $(1\,2\,3)$ and $(1\,3\,2)$ are not conjugate in A_4.

Hint: Apply 2(b) with

$$H = \{1,\ (1\,2)(3\,4),\ (1\,3)(2\,4),\ (1\,4)(2\,3)\}.$$

Solution: It is straightforward to check that H is a subgroup: the square of any element is 1, and the product of any two distinct non-identity elements is the other non-identity element. Next, H is normal, because conjugation preserves cycle types, and there are no other elements of A_4 (or indeed of S_4) of the same cycle type. The index of H in G is $|G|/|H| = 12/4 = 3$. By 2(b), if $(1\,2\,3)$ were conjugate to $(1\,3\,2) = (1\,2\,3)^{-1}$ then we would have $(1\,2\,3) \in H$; but clearly we do not.

(b) Show that $(1\,2\,3)$ and $(1\,3\,2)$ are conjugate in S_4.

Solution: Let $\sigma = (1\,2\,3)$ and $\tau = (1\,2)$, or $(1\,3)$, or $(2\,3)$. In any case we have $\tau\sigma\tau^{-1} = (1\,3\,2)$.

(c) Show that $(1\,2\,3)$ and $(1\,3\,2)$ are conjugate in A_5.

Solution: Let $\sigma = (1\,2\,3)$ and $\tau = (1\,2)(4\,5)$, or $(1\,3)(4\,5)$, or $(2\,3)(4\,5)$. The $(4\,5)$ commutes with σ, so again we have $\tau\sigma\tau^{-1} = (1\,3\,2)$. Since τ is a product of an even number of 2-cycles, we see that $\tau \in A_5$.