Midterm 2

Friday, March 3, 2017

1. (a) Show that if G is a group and $H \subset G$ is a subgroup of index 2 then H is normal.

   (b) Give an example of a group G and subgroup $H \subset G$ of index 3 that is *not* normal.

   (c) Give an example of a group G and subgroup $H \subset G$ of index 3 that *is* normal. Describe the quotient.

2. (a) Let G be a group, let A be an Abelian group, and let $\varphi\colon G \to A$ be a homomorphism. Show that if two elements $x, y \in G$ are conjugate then $\varphi(x) = \varphi(y)$.

   (b) Let G be a group, let $H \subset G$ be a *normal* subgroup of index 3, and let $g \in G$. Show that g is conjugate to g^{-1} then $g \in H$.
   Hint: Observe that $G/H \cong \mathbb{Z}/3$, and apply part (a) to the natural projection $\varphi\colon G \to G/H$.

   (c) Give an example of a group G, a subgroup $H \subset G$ of index 3 that is *not* normal, and an element $g \in G$ such that g is conjugate to g^{-1} but $g \notin H$.

3. (a) Show that $(1\,2\,3)$ and $(1\,3\,2)$ are not conjugate in A_4.
   Hint: Apply 2(b) with
   $$H = \{1,\ (1\,2)(3\,4),\ (1\,3)(2\,4),\ (1\,4)(2\,3)\}.$$

   (b) Show that $(1\,2\,3)$ and $(1\,3\,2)$ are conjugate in S_4.

   (c) Show that $(1\,2\,3)$ and $(1\,3\,2)$ are conjugate in A_5.