Solutions to Midterm 1

1. (a) Let G be a group. Show that if $g^2 = 1$ for every $g \in G$ then G is Abelian.

   Solution: Let $g, h \in G$ be given. Then $(gh)^2 = 1$, so $ghgh = 1$. Multiply on the left by g and on the right by h to get $g^2hgh^2 = gh$. But $g^2 = h^2 = 1$, so $hg = gh$.

   (b) Give an example of an Abelian group G which does not satisfy $g^2 = 1$ for every $g \in G$.

   Solution: The additive group of $\mathbb{Z}/3$.

   (c) Give an example of a non-Abelian group.

   Solution: The symmetric group S_3. We have $(1\,2)(1\,3) = (1\,3\,2)$, but $(1\,3)(2\,3) = (1\,2\,3)$.

2. Let F be the field $\mathbb{Z}/2 = \{0, 1\}$, and let $G = \mathrm{GL}_2(F)$ be the group of 2×2 matrices with entries in F and determinant $\neq 0$.

   (a) List the elements of G. Hint: The two columns have to be different, and neither can be zero.

   Solution:

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \quad \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix} \quad \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} \quad \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}.$$

   (b) Let G act on the vector space $V = F^2$ in the usual way: matrix times column vector. Describe the orbit and stabilizer of the vector $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$.

   Solution: The orbit has three elements, given by the first columns of the matrices above:

$$\begin{pmatrix}1\\ 0\end{pmatrix} \qquad \begin{pmatrix}0\\ 1\end{pmatrix} \qquad \begin{pmatrix}1\\ 1\end{pmatrix}.$$

   The stabilizer has two elements, given by the matrices with first column $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$:

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}.$$

   This agrees with $|G| = 6$.

   (c) Use this action to show that $G \cong S_3$.

   Solution: The orbit of $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ has three elements, giving a homomorphism $G \to S_3$. You could argue that it is injective, because if a matrix A fixes all non-zero vectors then it is the identity. Or you could argue that it is surjective, because the image contains all transpositions:

$$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \mapsto (1\,2) \qquad \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} \mapsto (1\,3) \qquad \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \mapsto (2\,3).$$

3. Continue to let $F = \mathbb{Z}/2$, and now let $G = \mathrm{GL}_3(F)$ act on $V = F^3$. We will use this action to compute the order of G.

(a) Show that the orbit of $\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$ is $V \setminus \{0\}$.

Solution: Let $v = \left(\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right) \in V \setminus \{0\}$ be given. If $x \neq 0$, then the matrix

$$A = \begin{pmatrix} x & 0 & 0 \\ y & 1 & 0 \\ z & 0 & 1 \end{pmatrix}$$

has determinant $x \neq 0$ and takes $\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$ to $\left(\begin{smallmatrix}x\\y\\z\end{smallmatrix}\right)$. If $y \neq 0$, take

$$A = \begin{pmatrix} x & 1 & 0 \\ y & 0 & 0 \\ z & 0 & 1 \end{pmatrix}.$$

If $z \neq 0$, take

$$A = \begin{pmatrix} x & 1 & 0 \\ y & 0 & 1 \\ z & 0 & 0 \end{pmatrix}.$$

(b) How many elements does $V \setminus \{0\}$ have?

Solution: V has $2^3 = 8$ elements, so $V \setminus \{0\}$ has 7 elements.

(c) Describe the stabilizer of $\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$, and find its order.

Solution: The stabilizer consists of invertible matrices with first column $\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)$, i.e. matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & c & d \\ 0 & e & f \end{pmatrix}$$

with a, b arbitrary and $\begin{pmatrix} c & d \\ e & f \end{pmatrix} \in \mathrm{GL}_2(F)$. For a we have 2 choices, for b we have 2 choices, and for the 2×2 block we have 6 choices as we saw in the previous problem. Thus the stabilizer has order 24.

(d) Use the orbit-stabilizer theorem to find $|G|$.

Solution: $7 \cdot 24 = 168$.