Homework 7

Due Wednesday, March 1, 2017

1. In lecture we showed that every group of order 9 is Abelian. Write the proof in your own words, according to your own taste.

   Optional: Do p^2 rather than 9.

2. Let G be the group of order 12 from Homework 4, problem 6:
$$\begin{aligned}\{\ 1,\ a,\ \ a^2,\ \ a^3,\ \ a^4,\ \ a^5,\\ b,\ ba,\ ba^2,\ ba^3,\ ba^4,\ ba^5\ \},\end{aligned}$$
   subject to the relations
$$a^6 = 1 \qquad\qquad b^2 = a^3 \qquad\qquad ab = ba^{-1}.$$
   Describe the conjugacy classes and write the class equation for G.

   Hint: First compute aga^{-1} for all $g \in G$, then compute bgb^{-1} for all $g \in G$. You don't need to compute any more than this.

   Further Hint: To compute aga^{-1} for all $g \in G$, first compute aba^{-1}, then exploit the fact that conjugating by a is a group homomorphism: $a(gh)a^{-1} = (aga^{-1})(aha^{-1})$. Similarly, to compute bgb^{-1}, first compute bab^{-1}.

3. Let $\sigma = (1\,2\,3\,4\,5) \in S_5$.
   (a) Find a $\tau \in S_5$ such that $\tau\sigma\tau^{-1} = \sigma^2$.
   (b) Find a $\tau \in S_5$ such that $\tau\sigma\tau^{-1} = \sigma^3$.
   (c) Find a $\tau \in S_5$ such that $\tau\sigma\tau^{-1} = \sigma^4$.

4. Very optional: Let $G = \mathrm{GL}_2(\mathbb{C})$. Show that every $A \in G$ is conjugate to exactly one of the following:
$$\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix},\ \lambda \neq \mu \qquad\qquad \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} \qquad\qquad \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}.$$
   Hint: Because $\mathbb{C}$ is algebraically closed, every $A \in G$ has at least one eigenvector.