Solutions to Homework 6

1. Describe all homomorphisms $D_4 \to \mathbb{Z}_2$. Where do they send r and s?

   Solution: There is of course the trivial homomorphism $\varphi\colon D_4 \to \mathbb{Z}_2$ with $\varphi(r) = \varphi(s) = 0$.

   In lecture we have seen that for any group G, non-trivial homomorphisms $G \to \mathbb{Z}_2$ are in bijection with subgroups of index 2: a homomorphism φ corresponds to the subgroup $\ker\varphi \subset G$. On homework 4, problem 5 we saw that D_4 has three subgroups of order 4:

   - $H = \langle r \rangle = \{1, r, r^2, r^3\}$. The homomorphism $\varphi\colon D_4 \to \mathbb{Z}_2$ with $\ker\varphi = H$ is determined by $\varphi(r) = 0$ and $\varphi(s) = 1$.
   - $H = \langle s, r^2 \rangle = \{1, s, r^2, sr^2\}$. The homomorphism $\varphi\colon D_4 \to \mathbb{Z}_2$ with $\ker\varphi = H$ is determined by $\varphi(r) = 1$ and $\varphi(s) = 0$.
   - $H = \langle sr, r^2 \rangle = \{1, sr, r^2, sr^2\}$. The homomorphism $\varphi\colon D_4 \to \mathbb{Z}_2$ with $\ker\varphi = H$ is determined by $\varphi(r) = 1$ and $\varphi(s) = 1$.

2. For each $v \in \mathbb{Z} \times \mathbb{Z}$, find an integer k such that $(\mathbb{Z} \times \mathbb{Z})/v \cong \mathbb{Z} \times \mathbb{Z}_k$. (Prove it using the first isomorphism theorem.)

   (a) $v = (2, 2)$.

   Solution: $k = 2$. Consider the homomorphism $\varphi\colon \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}_2$ given by $\varphi(x, y) = (x - y, x \pmod 2)$. First, φ is surjective: for any $z \in \mathbb{Z}$ we have $\varphi(z, 0) = (z, 0)$ and $\varphi(z + 1, 1) = (z, 1)$. Second, $v \in \ker\varphi$: indeed, we have $\varphi(2, 2) = (0, 0)$. Third, if $w \in \ker\varphi$ then $w = nv$ for some $n \in \mathbb{Z}$: write $w = (x, y)$; then $x \equiv 0 \pmod 2$, so we can write $x = 2n$ for some $n \in \mathbb{Z}$; and $x - y = 0$, so $y = x = 2n$; thus $w = nv$.

   (b) $v = (2, 3)$.

   Solution: $k = 1$; or in other words, $(\mathbb{Z}\times\mathbb{Z})/v \cong \mathbb{Z}$. Consider the homomorphism $\varphi\colon \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$ given by $\varphi(x, y) = 3x - 2y$. First, φ is surjective: for an even number $2z \in \mathbb{Z}$ we have $\varphi(0, -z) = 2z$, and for an odd number $2z + 1 \in \mathbb{Z}$ we have $\varphi(1, 1 - z) = 2z + 1$.

Second, $v \in \ker\varphi$: indeed, we have $\varphi(2,3) = 6 - 6 = 0$. Third, if $w \in \ker\varphi$ then $w = nv$ for some $n \in \mathbb{Z}$: write $w = (x, y)$; then $3x - 2y = 0$, so $3x = 2y$. Then 2 divides $3x$, so 2 divides x, so we can write $x = 2n$ for some $n \in \mathbb{Z}$. Then we have $6n = 2y$, so $y = 3n$. Thus $w = nv$.

(c) $v = (2, 4)$.

Solution: $k = 2$. Consider the homomorphism $\varphi\colon \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}_2$ given by $\varphi(x, y) = (2x - y, x \pmod 2)$. First, φ is surjective: for any $z \in \mathbb{Z}$ we have $\varphi(0, -z) = (z, 0)$ and $\varphi(1, 2 - z) = (z, 1)$. Second, $v \in \ker\varphi$: indeed, we have $\varphi(2, 4) = (0, 0)$. Third, if $w \in \ker\varphi$ then $w = nv$ for some $n \in \mathbb{Z}$: write $w = (x, y)$; $x \equiv 0 \pmod 2$, so we can write $x = 2n$ for some $n \in \mathbb{Z}$; and $2x - y = 0$, so $y = 2x = 4n$; thus $w = nv$.

(d) Optional: An arbitrary $v = (m, n)$.

Solution: I claim that $k = \gcd(m, n)$ works. Let $a = m/k$ and $b = n/k$. Then $\gcd(a, b) = 1$, so we can write $ad - bc = 1$ for some $c, d \in \mathbb{Z}$ by the Euclidean algorithm. Now consider the homomorphism $\varphi\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_k$ given by

$$\varphi(x, y) = (ay - bx, dx - cy \pmod k).$$

First, φ is surjective: we have $\varphi(c, d) = (1, 0)$ and $\varphi(a, b) = (0, 1)$, and these generate $\mathbb{Z} \times \mathbb{Z}_k$. Second, $v \in \ker\varphi$: we have $\varphi(m, n) = \varphi(ak, bk) = (0, k) = (0, 0)$. Third, if $w \in \ker\varphi$ then w is a multiple of v: write $w = (x, y)$; then $ay - bx = 0$, so $ay = bx$. Then a divides bx, so a divides x (because a and b are relatively prime), so we can write $x = al$ for some $l \in \mathbb{Z}$. Then $ay = abl$, so $y = bl$. Next, we know that k divides $dx - cy = adl - bcl = l$, so l is a multiple of k, so $w = (al, bl)$ is a multiple of $v = (ak, bk)$.

3. Let G be a group, and let $H, N \subset G$ be subgroups.

(a) Show that $H \cap N$ is a subgroup.

Solution: Let $x, y \in H \cap N$. Because H is a subgroup, we have $xy \in H$ and $x^{-1} \in H$. Because N is a subgroup, we have $xy \in N$ and $x^{-1} \in N$. Thus $xy \in H \cap N$ and $x^{-1} \in H \cap N$. Thus $H \cap N$ is a subgroup.

(b) Give an example to show that $H \cup N$ need not be a subgroup.

Solution: Let $G = S_3$, let $H = \{1, (1\,2)\}$, and let $N = \{1, (1\,3)\}$. Then $(1\,2) \cdot (1\,3) = (1\,3\,2) \notin H \cup N$, so $H \cup N$ is not a subgroup.

(c) Give an example to show that

$$HN = \{hn : h \in H, n \in N\} \subset G$$

need not be a subgroup.

Solution: Take the same G, H, and N as in the previous part. Then

$$HN = \{1, (1\,2), (1\,3), (1\,3\,2)\}.$$

But $(1\,3) \cdot (1\,2) = (1\,2\,3) \notin HN$, so HN is not a subgroup.

(d) Show that if N is normal in G then HN is a subgroup.

Solution: Let $h_1 n_1$ and $h_2 n_2$ be two elements of HN. Then

$$h_1 n_1 h_2 n_2 = h_1 (h_2 h_2^{-1}) n_1 h_2 n_2 = h_1 h_2 (h_2^{-1} n_1 h_2) n_2.$$

Because N is normal, $h_2^{-1} n_1 h_2 \in N$. So we have $h_1 h_2 \in H$ and $(h_2^{-1} n_1 h_2) n_2 \in N$, so the product above is in HN.

Next, we have

$$(h_1 n_1)^{-1} = n_1^{-1} h_1 = h_1 (h_1^{-1} n_1^{-1} h_1).$$

Because $n_1^{-1} \in N$ and N is normal, we have $h_1^{-1} n_1^{-1} h_1 \in N$, so the product above is in HN.

(e) Show that if N is normal in G then $H \cap N$ is normal in H.

Solution: We must show that for all $x \in H \cap N$ and $h \in H$, we have $hxh^{-1} \in H \cap N$. First, $hxh^{-1} \in H$ because $x \in H$ and H is a subgroup. Second, $hxh^{-1} \in N$ because $x \in N$ and N is normal. Thus $hxh^{-1} \in H \cap N$ as desired.

4. Optional: Let G be a group, and let $\Delta \subset G \times G$ be the "diagonal" subgroup consisting of elements of the form (g, g). Show that G is Abelian if and only if Δ is a normal subgroup.

Solution: If G is Abelian then $G \times G$ is Abelian, and every subgroup of an Abelian group is normal.

For the converse, let $x, y \in G$, and consider the elements $(xy, xy) \in \Delta$ and $(1, y) \in G \times G$. Then

$$(1, y)(xy, xy)(1, y)^{-1} = (xy, yxyy^{-1}) = (xy, yx).$$

If Δ is normal then $(xy, yx) \in \Delta$, so $xy = yx$.