Homework 6

Due Wednesday, February 22, 2017

1. Describe all homomorphisms $D_4 \to \mathbb{Z}_2$. Where do they send r and s?

   Hint: This question is secretly about subgroups of index 2.

2. For each $v \in \mathbb{Z} \times \mathbb{Z}$, find an integer k such that $(\mathbb{Z} \times \mathbb{Z})/v \cong \mathbb{Z} \times \mathbb{Z}_k$. (Prove it using the first isomorphism theorem.)

   (a) $v = (2, 2)$.

   (b) $v = (2, 3)$.

   (c) $v = (2, 4)$.

   (d) Optional: An arbitrary $v = (m, n)$.

3. Let G be a group, and let $H, N \subset G$ be subgroups.

   (a) Show that $H \cap N$ is a subgroup.

   (b) Give an example to show that $H \cup N$ need not be a subgroup.

   (c) Give an example to show that

   $$HN = \{hn : h \in H, n \in N\} \subset G$$

   need not be a subgroup.

   Hint: You can find an example in $G = S_3$.

   (d) Show that if N is normal in G then HN is a subgroup.

   (e) Show that if N is normal in G then $H \cap N$ is normal in H.

4. Optional: Let G be a group, and let $\Delta \subset G \times G$ be the "diagonal" subgroup consisting of elements of the form (g, g). Show that G is Abelian if and only if Δ is a normal subgroup.