Homework 5

Due Wednesday, February 15, 2017

1. Let G be a group and H a subgroup. For $g, g' \in G$, show that the following are equivalent:

   (a) $gH = g'H$.

   (b) $gH \subset g'H$.

   (c) $g \in g'H$.

   (d) $gH \cap g'H \neq \emptyset$.

   (e) $g^{-1}g' \in H$.

   Warning: Do not assume that G or H is finite.

2. Let G be a group and H a subgroup. Show that the following are equivalent:

   (a) For all $g \in G$ one has $gH = Hg$.

   (b) For all $g \in G$ one has $gH \subset Hg$.

   (c) For all $g \in G$ one has $gHg^{-1} = H$.

   (d) For all $g \in G$ one has $gHg^{-1} \subset H$.

   Notice the difference between the statements

   $$\forall g \in G, (gH \subset Hg \Rightarrow gH = Hg)$$

   and

   $$(\forall g \in G, gH \subset Hg) \Rightarrow (\forall g \in G, gH = Hg).$$

   The implication (b) $\Rightarrow$ (a) is the latter. The former holds if H is finite. Optional: Does it hold if H is infinite?

3. (a) Let a group G act on a set X (on the left), and let $H \subset G$ be the stabilizer of a point $x \in X$. Show that for $g \in G$, the stabilizer of gx is gHg^{-1}.

   (b) Give an explicit example where $G = D_4$.

4. (a) Let $\varphi\colon G \to H$ be a surjective group homomorphism, and let $K = \ker \varphi$. Show that for all $h \in H$, the fiber

$$\varphi^{-1}(h) = \{g \in G : \varphi(g) = h\}$$

   is a coset of K. (Left or right doesn't matter, because K is normal. As always, beware that φ^{-1} is not a map $H \to G$.)

   (b) For the homomorphism $\varphi\colon \mathbb{C}^\times \to \mathbb{R}$ given by $\varphi(z) = \log |z|$, describe the kernel and its cosets geometrically.

   (c) For the homomorphism $\varphi\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $\varphi(x, y) = x + y$, describe the kernel and its cosets geometrically.

5. Let G be a group of order 8, and suppose that $g^2 = 1$ for all $g \in G$. Show that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

6. Optional: Show that

   (a) $D_6 \cong D_3 \times \mathbb{Z}_2$.

   (b) $D_8 \not\cong D_4 \times \mathbb{Z}_2$.

   (c) Generalize to D_{2n}.