Solutions to Homework 3

1. Let G be a group. Show that G is Abelian if and only if the map $\varphi\colon G \to G$ given by $\varphi(g) = g^2$ is a homomorphism.

   Solution: If G is Abelian then for any $g, h \in G$ we have

   $$\varphi(gh) = (gh)^2 = ghgh = gghh = g^2h^2 = \varphi(g)\varphi(h).$$

   Conversely, if φ is a homomorphism then for any $g, h \in G$ we have

   $$(gh)^2 = g^2h^2.$$

   Multiplying by g^{-1} on the left and h^{-1} on the right we get

   $$hg = gh.$$

2. Let G be a group, let X be a set, and let S_X denote the group of bijections $X \to X$ under composition.

   (a) Suppose that G acts on X. For $g \in G$, let $l_g\colon X \to X$ be the map $l_g(x) = g \cdot x$. Show that $l_g \in S_X$, and that the map $G \to S_X$ given by $g \mapsto l_g$ is a homomorphism.

   Solution: First we argue that $l_g\colon X \to X$ is bijective. It is injective: if $l_g(x) = l_g(y)$ then $gx = gy$; acting on the right with g^{-1} we get $g^{-1}(gx) = g^{-1}(gy)$, so $(g^{-1}g)x = (g^{-1}g)y$, so $1 \cdot x = 1 \cdot y$, so $x = y$. It is surjective: for any $x \in X$, we have $l_g(g^{-1}x) = g(g^{-1}x) = (gg^{-1})x = 1 \cdot x = x$.

   Next we argue that the map $g \mapsto l_g$ is a homomorphism. Let $g, h \in G$ be given. Then for any $x \in X$ we have $l_g(l_h(x)) = g(hx) = (gh)x = l_{gh}(x)$. Thus $l_g \circ l_h = l_{gh}$.

   (b) Conversely, suppose that that $\varphi\colon G \to S_X$ is a homomorphism. Show that $g \cdot x = \varphi(g)(x)$ defines an action of G on X.

   Solution: For the sake of clarity we remark that $\varphi(g)$ is an element of S_X, and in particular a map $X \to X$, and $\varphi(g)(x)$ is the element of X gotten by applying the map $\varphi(g)$ to x.

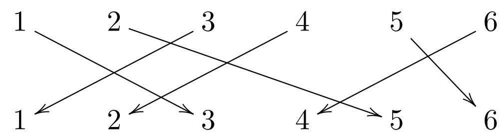
First we have

$$\begin{aligned} g \cdot (h \cdot x) = \varphi(g)(\varphi(h)(x)) = (\varphi(g) \circ \varphi(h))(x) \\ = \varphi(gh)(x) = (gh) \cdot x, \end{aligned}$$

where the third equality is because φ is a homomorphism. Next, because φ is a homomorphism, $\varphi(1)$ is the identity map, so

$$1 \cdot x = \varphi(1)(x) = x.$$

3. Let σ be the following elements of S_6:



Write σ, σ^2, and σ^{-1} in cycle notation.

Solution:

$$\begin{aligned} \sigma &= (1\,3)(2\,5\,6\,4) \\ \sigma^2 &= (2\,6)(4\,5) \\ \sigma^{-1} &= (1\,3)(2\,4\,6\,5) \end{aligned}$$

4. In S_6, let

$$\sigma = (1\,6\,2)(3\,4) \qquad\qquad \tau = (3\,4\,5\,6).$$

Compute $\sigma \cdot \tau$, $\tau \cdot \sigma$, σ^2, and τ^2.

Solution:

$$\begin{aligned} \sigma \cdot \tau &= (1\,6\,4\,5\,2) \\ \tau \cdot \sigma &= (1\,3\,5\,6\,2) \\ \sigma^2 &= (1\,2\,6) \\ \tau^2 &= (3\,5)(4\,6) \end{aligned}$$

5. Let $\sigma = (1\,2\,3\,4\,5\,6) \in S_6$. For which integers n is σ^n a 6-cycle?

Solution: We have

$$\begin{aligned}
\sigma &= (1\,2\,3\,4\,5\,6)\\
\sigma^2 &= (1\,3\,5)(2\,4\,6)\\
\sigma^3 &= (1\,4)(2\,5)(3,6)\\
\sigma^4 &= (1\,5\,3)(2\,6\,4)\\
\sigma^5 &= (1\,6\,5\,4\,3\,2)\\
\sigma^6 &= 1.
\end{aligned}$$

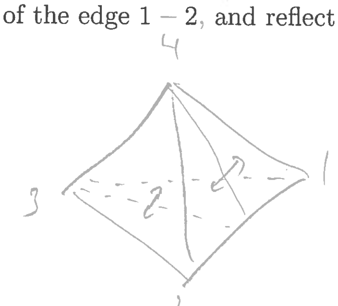
Thus σ^n is a 6-cycle if and only if $n \equiv 1$ or $5 \mod 6$.

In general, if τ is an m-cycle, you can check that τ^m is an m-cycle if and only if $\gcd(m, n) = 1$.

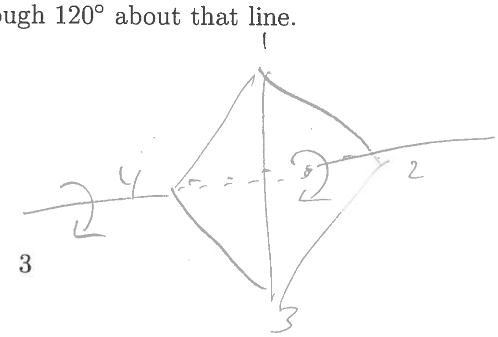
6. Let G be the symmetry group of the regular tetrahedron. In lecture we saw that the action of G on the four vertices induces an isomorphism $G \to S_4$. Draw pictures of the symmetries corresponding to the following elements of S_4:

$$(1\,2) \qquad (1\,2\,3) \qquad (1\,3\,2) \qquad (1\,2)(3\,4) \qquad (1\,2\,3\,4)$$

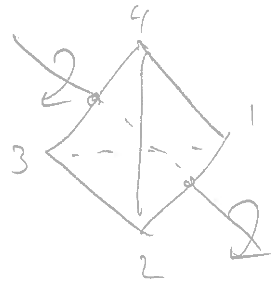
Solution: For $(1\,2)$, we take the plane spanned by vertex 3, vertex 4, and the midpoint of the edge $1 - 2$, and reflect across that plane:



For $(1\,2\,3)$, we take the line joining vertex 4 with the center of the opposite face, and rotate through 120° about that line.



For $(1\,2)(3\,4)$, we take the line joining the midpoint of the edge $1-2$ and the midpoint of the edge $3-4$, and rotate through $180°$ about that line.



The 4-cycle $(1\,2\,3\,4)$ is perhaps impossible to draw: it is not a rotation, because the associated 3×3 matrix has negative determinant; nor is it a reflection, because it has order 4 rather than 2. You could describe it as the rotation $(1\,2\,3)$ followed by the reflection $(1\,4)$, or as the rotation $(1\,2)(3\,4)$ followed by the reflection $(1\,3)$, or as the composition of three reflections. Another option is to do the antipodal map on $\mathbb{R}^3$, taking $(x, y, z) \to (-x, -y, -z)$, which does not fix the tetrahedron, and then rotate the tetrahedron a bit to take 1 to 2, 2 to 3, etc.

7. Let the symmetric group S_3 act on the vector space $\mathbb{R}^3$ by permuting the three coordinates.

   (a) Describe the orbit and stabilizer of the point $(4, 5, 6) \in \mathbb{R}^3$.
   Solution: The orbit has six elements:

   $$(4,5,6),\ (4,6,5),\ (5,4,6),\ (5,6,4),\ (6,4,5),\ (6,5,4).$$

   The stabilizer is the trivial subgroup of S_3.

   (b) Describe the orbit and stabilizer of the point $(4, 5, 5) \in \mathbb{R}^3$.
   Solution: The orbit has three elements:

   $$\{(4,5,5),(5,4,5),(5,5,4).$$

   The stabilizer consists of the identity and the transposition $(2\,3)$.

   (c) Optional: Show that (x, y, z) and (x', y', z') lie in the same orbit if and only if

   $$\begin{aligned} x + y + z &= x' + y' + z', \\ xy + xz + yz &= x'y' + x'z' + y'z', \\ xyz &= x'y'z'. \end{aligned}$$

Solution: Clearly the three polynomials are invariant under permutation of x, y, z: for example, if we switch y and z then

$$xz + xy + zy = xy + xz + yz.$$

The interesting part is the converse. For legibility let us write a, b, c rather than x', y', z'. Consider

$$\begin{aligned}&(x-a)(x-b)(x-c)\\&\qquad\qquad = x^3 - (a+b+c)x^2 + (ab+ac+bx)x - abc.\end{aligned}$$

If $x+y+z = a+b+c$ etc. then this equals

$$\begin{aligned}&x^3 - (x+y+z)x + (xy+xz+yz)x - xyz\\&\qquad\qquad\qquad\qquad = (x-x)(x-y)(x-z) = 0.\end{aligned}$$

Thus $x - a = 0$, or $x - b = 0$, or $x - c = 0$. Suppose first that $x = a$; then the equations become

$$\begin{aligned}x+y+z &= x+b+c\\ xy+xz+yz &= xb+xc+bc\\ xyz &= xbc.\end{aligned}$$

Subtracting x in the first equation we get

$$y + z = b + c.$$

If $x = 0$ then the second equation becomes

$$yz = bc,$$

and if $x \neq 0$ then we can divide by x in the third equation to get $yz = bc$ again. Now we consider

$$\begin{aligned}&(y-b)(y-c) = y^2 - (b+c)y + bc\\&\qquad\qquad = y^2 - (y+z)y + yz = (y-y)(y-z) = 0,\end{aligned}$$

so $y = b$ or $y = c$, and from $y + z = b + c$ we find that $z = c$ or $z = b$. In the first case we take $\sigma = 1$; in the second we take $\sigma = (2\,3)$.

Simlarly, if $x = b$ then we find that either $y = a$ and $z = c$, in which case we take $\sigma = (1\,2)$, or $y = c$ and $z = a$, in which case we take $\sigma = (1\,2\,3)$. If $x = c$ then either $y = a$ and $z = b$, in which case we take $\sigma = (1\,3\,2)$, or $y = b$ and $z = a$, in which case we take $\sigma = (1\,3)$.

Observe that the same proof would work in any integral domain.