Homework 3

Due Wednesday, February 1, 2017

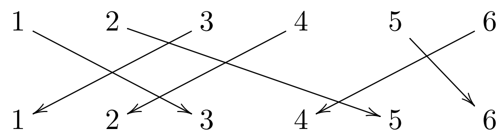
1. Let G be a group. Show that G is Abelian if and only if the map $\varphi\colon G \to G$ given by $\varphi(g) = g^2$ is a homomorphism.

2. Let G be a group, let X be a set, and let S_X denote the group of bijections $X \to X$ under composition.

   (a) Suppose that G acts on X. For $g \in G$, let $l_g\colon X \to X$ be the map $l_g(x) = g \cdot x$. Show that $l_g \in S_X$, and that the map $G \to S_X$ given by $g \mapsto l_g$ is a homomorphism.

   (b) Conversely, suppose that that $\varphi\colon G \to S_X$ is a homomorphism. Show that $g \cdot x = \varphi(g)(x)$ defines an action of G on X.

3. Let σ be the following elements of S_6:



   Write σ, σ^2, and σ^{-1} in cycle notation.

4. In S_6, let

$$\sigma = (1\,6\,2)(3\,4) \qquad\qquad \tau = (3\,4\,5\,6).$$

   Compute $\sigma \cdot \tau$, $\tau \cdot \sigma$, σ^2, and τ^2.

5. Let $\sigma = (1\,2\,3\,4\,5\,6) \in S_6$. For which integers n is σ^n a 6-cycle?

6. Let G be the symmetry group of the regular tetrahedron. In lecture we saw that the action of G on the four vertices induces an isomorphism $G \to S_4$. Draw pictures of the symmetries corresponding to the following elements of S_4:

$$(1\,2) \qquad (1\,2\,3) \qquad (1\,3\,2) \qquad (1\,2)(3\,4) \qquad (1\,2\,3\,4)$$

7. Let the symmetric group S_3 act on the vector space $\mathbb{R}^3$ by permuting the three coordinates.

   (a) Describe the orbit and stabilizer of the point $(4, 5, 6) \in \mathbb{R}^3$.

   (b) Describe the orbit and stabilizer of the point $(4, 5, 5) \in \mathbb{R}^3$.

   (c) Optional: Show that (x, y, z) and (x', y', z') lie in the same orbit if and only if

$$\begin{aligned} x + y + z &= x' + y' + z', \\ xy + xz + yz &= x'y' + x'z' + y'z', \\ xyz &= x'y'z'. \end{aligned}$$