Final Exam

Thursday, March 23, 2017

1. Let G be a group of order $21 = 3 \cdot 7$.

   (a) Use the Sylow theorems to show that G contains an element x of order 7, and an element y of order 3, and that $yxy^{-1} = x^k$ for some integer k.

   (b) Show that every element of G can be written uniquely as $x^i y^j$, for some $i \in \{0, 1, 2, \ldots, 6\}$ and $j \in \{0, 1, 2\}$.

   Hint: Let $H = \langle x \rangle$ and $K = \langle y \rangle$, and argue that $H \cap K = \{1\}$. Then argue that $G = H \cup Hy \cup Hy^2$, or if you prefer, $G = K \cup xK \cup x^2K \cup \cdots \cup x^6K$.

   (c) Show that $k^3 \equiv 1 \pmod 7$. Which values of k satisfy this equation?

   Hint: There are three solutions (mod 7).

   (d) If $k = 1$, show that the map $\varphi\colon \mathbb{Z}_7 \times \mathbb{Z}_3 \to G$ given by $\varphi(i, j) = x^i y^j$ is an isomorphism.

   Hint: Don't spend time arguing that φ is bijective; you already did that in part (b).

   (e) Let $T \subset \mathrm{GL}_2(\mathbb{Z}_7)$ be the set of matrices of the form

   $$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

   with $a \in \{1, 2, 4\}$ and $b \in \{0, 1, 2, \ldots, 6\}$. Show that T is a subgroup of $\mathrm{GL}_2(\mathbb{Z}_7)$.

   (f) Show that the matrix

   $$x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in T$$

   satisfies $x^7 = 1$.

   For each of the two interesting values of k in part (c), find a matrix $y \in T$ with $y^3 = 1$ and $yx = x^k y$.

   Hint: In both cases you can take y to be a diagonal matrix.

Thus you have shown that a group of order 21 is either isomorphic to $\mathbb{Z}_7 \times \mathbb{Z}_3 \cong \mathbb{Z}_{21}$, or to T. By our earlier study of groups of order 9 and 15, we see that T is the smallest non-Abelian group of odd order.

2. Consider $\mathbb{Z}_2$ as a field with elements $\{0, 1\}$. Let G be the group of 3×3 upper triangular matrices with entries in $\mathbb{Z}_2$:

$$\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix},$$

with implied zeroes below the diagonal.

(a) For each of the eight elements $g \in G$, compute g^2. Circle the elements of order 2, and put a box around the elements of order 4.

Hint: Since $|G| = 8$, every element has order 1, 2, or 4, so computing g^2 is enough; you don't have to compute g^3 or g^4.

(b) We have seen that there are two non-Abelian groups of order 8, up to isomorphism: the dihedral group D_4 and the quaternion group Q. Show that our group $G \ncong Q$. (So we must have $G \cong D_4$.)

(c) Choose an element $r \in G$ of order 4, and an element $s \in G$ of order 2 that is different from r^2. Verify that $rs = sr^3$.

Hint: You don't have to compute r^3; just notice that it must be the other element of order 4.

(d) Let G act on the set of column vectors $(\mathbb{Z}_2)^3$ by left multiplication. The orbit of $\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$ has four elements. Write them in a square, so that r acts as a clockwise rotation. Does s act on this square by reflection left-to-right, top-to-bottom, or diagonally?

Hint: Write $\left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$ in one corner, multiply by r, write the result in the next corner, multiply by r again, etc.

I wanted to include the following problem, but the exam was getting too long. Of course I would have broken it into many parts.

3. In lecture we showed that the alternating group A_n has no non-trivial normal subgroups for $n \geq 5$. Use this to show that the only non-trivial normal subgroup of S_n is A_n, again for $n \geq 5$.