Homework 8

Due Wednesday, November 23, 2016

1. Show that the ideals (y) and $(x, y) \subset \mathbb{Q}[x, y]$ is prime.

2. Let R be the ring of continuous functions on $\mathbb{R}$. Show that the ideal $I = \{f \in R : f(t) = 0 \text{ for all } t \in [0, 1]\}$ is not prime.

3. Let $R = \mathbb{Z}[\sqrt{-3}]$, and consider the elements $\sqrt{-3}$, $1 + \sqrt{-3}$, and $2 + \sqrt{-3}$ in R. In each case, show that $R/r \cong \mathbb{Z}/k$ for some integer k. Deduce that two of these elements are prime and the other is not.

4. Let R be a commutative ring, let $I, J \subset R$ be ideals, and let $P \subset R$ be a prime ideal. Show that if $IJ \subset P$ then $I \subset P$ or $J \subset P$.

5. In lecture, after giving a somewhat informal proof that every PID is a UFD, we said that a more formal proof should be broken into the following steps:

   (a) Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be a chain of ideals. Then
   $$I = I_1 \cup I_2 \cup I_3 \cup \cdots$$
   is an ideal.

   (b) Let R be a ring in which every ideal is finitely generated: that is, for every ideal $I \subset R$ there are $a_1, \ldots, a_n \in R$ such that $I = (a_1, \ldots, a_n)$. Then there is no infinite increasing chain of ideals $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$. Equivalently, any chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ eventually stabilizes: that is, there is a k such that $I_k = I_{k+1} = I_{k+2} = \cdots$. (The converse is also true. Such a ring is called *Noetherian*.)

   (c) Let R be a Noetherian ring. Then for every $r \in R$ there are irreducible elements $r_1, r_2, \ldots, r_m \in R$ such that $r = r_1 r_2 \cdots r_m$. (We do not make any claim about uniqueness of this factorization.)

   (d) If R is a principal ideal domain and $r \in R$ is irreducible, then r is prime.

(e) If R is an integral domain in which every irreducible is prime, factorizations are unique: that is, if we have $r_1 r_2 \cdots r_m = s_1 s_2 \cdots s_n$ with all r_i and s_i irreducible, then $m = n$, and we can reorder the s_i's so that there are units $u_1, u_2, \ldots, u_m$ with $s_i = u_i r_i$.

Write good, clean proofs of two of these statements.

6. What is one question you have about last week's lectures?