Homework 7

Due Wednesday, November 16, 2016

1. Here is a (partial) analogy between ideals and elements:

ideals	elements
$J \subset I$	$a \mid b$
$I + J = K$	$\dots$
$I \cap J = K$	$\dots$
$IJ = K$	$\dots$
I is prime	$\dots$
$I = J$	$\dots$
$I = R$	$\dots$

   (a) Fill in the rest of the table. For example, which one corresponds $\mathrm{lcm}(a, b) = c$?

   (b) To what extent can this analogy be made precise?
   Hint: Certainly something can be said in a principal ideal domain, and maybe a bit more generally.

2. Let R be a commutative ring. Show that $r \in R$ is prime if and only if R/r is an integral domain.

3. Show that for a prime $p \in \mathbb{Z}$, the following are equivalent:

   (a) p can be written as $x^2 + y^2$ for $x, y \in \mathbb{Z}$.

   (b) p is not prime in $\mathbb{Z}[i]$.

   (c) $\mathbb{Z}[i]/p$ is not an integral domain.

   (d) $\mathbb{Z}_p[x]/(x^2 + 1)$ is not an integral domain.
   Hint: Consider $\mathbb{Z}[x]/(p, x^2 + 1)$. In lecture we showed that for a commutative ring R and two elements $r_1, r_2 \in R$, we have $(R/r_1)/\bar{r}_2 \cong R/(r_1, r_2)$.

   (e) $x^2 + 1$ is not prime in $\mathbb{Z}/p[x]$.

(f) -1 is a square mod p: that is, there is an $n \in \mathbb{Z}$ with $n^2 \equiv -1 \pmod{p}$.

4. Postponed till next week.

5. What is one question you have about last week's lectures?