Homework 6

Due Wednesday, November 9, 2016

1. Let R be a commutative ring and let $I, J \subset R$ be ideals. Earlier you showed that $I + J$, IJ, and $I \cap J$ are ideals, and that $IJ \subset I \cap J$.
   (a) Show that if $I + J = (1)$ then $IJ = I \cap J$.
   (b) Show that the converse of part (a) does not hold in general by taking $R = \mathbb{Z}[x]$, $I = (2)$, and $J = (x)$.

2. Let R be a commutative ring, let $a, b, c, d \in R$, and consider the ideals $I = (a, b)$ and $J = (c, d)$.
   (a) Show that $I + J = (a, b, c, d)$.
   (b) Show that $IJ = (ac, ad, bc, bd)$.

3. Let $R = \mathbb{Z}[\sqrt{-5}]$, and consider the ideals

$$\begin{aligned} I_1 &= (2, 1 + \sqrt{-5}) \\ I_2 &= (2, 1 - \sqrt{-5}) \\ I_3 &= (1 + \sqrt{-5}, 1 - \sqrt{-5}) \\ J_1 &= (3, 1 + \sqrt{-5}) \\ J_2 &= (3, 1 - \sqrt{-5}). \end{aligned}$$

   (a) Show that $I_1 = I_2 = I_3$.
   (b) Show that $I_1 + J_1$, $I_1 + J_2$, and $J_1 + J_2$ are all equal to (1).
   (c) Find $J_1 J_2$, $I_1 J_1$, $I_2 J_2$, and $I_1 I_2$. Hint: They are all principal ideals, i.e. they can be generated by one element.

4. Let $I_1, J_1, J_2 \subset R$ be as in the previous problem.

   (a) Show that $R/I_1 \cong \mathbb{Z}/2$. Hint: In lecture, in week 4 or so when we first discussed ideals, we wrote down a homomorphism $\varphi\colon R \to \mathbb{Z}/2$ with $\ker\varphi = I_1$.

   (b) Show that $R/J_1 \cong \mathbb{Z}/3$ and $R/J_2 \cong \mathbb{Z}/3$.

   (c) Show that $R/(3) \cong \mathbb{Z}/3 \times \mathbb{Z}/3$.
   Hint: Use the Chinese Remainder Theorem.

5. What is one question you have about last week's lectures?