Solutions to Homework 4

1. Let R be a ring and $\varphi\colon \mathbb{Q} \to R$ be a homomorphism. Show that either $R = 0$ or φ is injective. Hint: $\ker\varphi$ is an ideal of $\mathbb{Q}$, and in lecture we described all of these.

   Solution: In lecture we saw that the only ideals of $\mathbb{Q}$ are (0) and (1). If $\ker\varphi = (0)$ then φ is injective. If $\ker\varphi = (1)$ then in particular $\varphi(1) = 0$; but we also have $\varphi(1) = 1$, so $1 = 0$ in R, so R is the zero ring.

2. In lecture we saw that every ideal in $\mathbb{Q}[x]$ is *principal*: that is, for every ideal $I \subset \mathbb{Q}[x]$ there is an $f \in \mathbb{Q}[x]$ such that $I = (f)$. In contrast, show that $(2, x) \subset \mathbb{Z}[x]$ is not a principal.

   Solution: Suppose that $(2, x) \subseteq (f)$ for some $f \in \mathbb{Z}[x]$. I claim that $f = \pm 1$. Since $2 \in (f)$, we can write $2 = fg$ for some $g \in \mathbb{Z}[x]$. Taking degrees, we get $0 = \deg f + \deg g$, so $\deg f = 0$, so f is a constant polynomial, say $f = a$ with $a \in \mathbb{Z}$. Since $x \in (f) = (a)$, we can write $x = ah$ for some $h \in \mathbb{Z}[x]$. Plugging in $x = 1$ we get $1 = a \cdot h(1)$, so a is a unit in $\mathbb{Z}$, that is, $a = \pm 1$. So if $(2, x) \subset (f)$ then $f = \pm 1$, as claimed. But then (f) is all of $\mathbb{Z}[x]$, and we have seen in lecture that $(2, x)$ is the set of polynomials whose constant term is even, which is not all of $\mathbb{Z}[x]$.

3. (a) Show that there is no homomorphism $\varphi\colon \mathbb{Q}[x] \to \mathbb{Z}[y]$.

   Solution: Suppose that there were such a φ, and let $f = \varphi(\frac{1}{2})$. Then $2f = f + f = \varphi(\frac{1}{2}) + \varphi(\frac{1}{2}) = \varphi(\frac{1}{2} + \frac{1}{2}) = \varphi(1) = 1$. Writing $f = a_0 + a_1 x + \cdots + a_n x^n$, we find that $2a_0 = 1$ and $2a_i = 0$ for $i > 0$. But there is no $a_0 \in \mathbb{Z}$ with $2a_0 = 1$.

   Alternatively, you could let $i\colon \mathbb{Q} \to \mathbb{Q}[x]$ be the inclusion of the constant polynomials; then $\varphi \circ i\colon \mathbb{Q} \to \mathbb{Z}[x]$ is injective by problem 1. We have seen in lecture that a homomorphism takes units to units. But there are infinitely many units in $\mathbb{Q}$, and we have seen that the units in $\mathbb{Z}[x]$ are just the constant polynomials ± 1, and an infinite set cannot inject into a set with two elements.

(b) Describe all homomorphisms $\psi\colon \mathbb{Z}[y] \to \mathbb{Q}[x]$.

Solution: I claim that the map ψ is determined by where it sends y. Indeed, we must have $\psi(0) = 0$, and $\psi(1) = 1$, and

$$\psi(2) = \psi(1+1) = \psi(1) + \psi(1) = 1 + 1 = 2,$$

and so on: $\psi(a) = a$ for all $a \in \mathbb{Z}$. Thus for an arbitrary

$$f = a_0 + a_1 y + a_2 y^2 + \cdots + a_n y^n \in \mathbb{Z}[y]$$

we have

$$\psi(f) = a_0 + a_1\psi(y) + a_2\psi(y)^2 + \cdots + a_n\psi(y)^n.$$

So $\psi(y)$ determines $\psi(f)$ for every $f \in \mathbb{Z}[y]$.

On the other hand, for any $g \in \mathbb{Q}[x]$, the map

$$\psi(a_0 + a_1 y + a_2 y^2 + \cdots + a_n y^n) = a_0 + a_1 g + a_2 g^2 + \cdots + a_n g^n$$

is a homomorphism. So there is exactly one homomorphism $\psi\colon \mathbb{Z}[y] \to \mathbb{Q}[x]$ for every $g \in \mathbb{Q}[x]$.

(c) Optional: Show that no such ψ is surjective.

Solution: Write $\psi(y) = b_0 + b_1 x + \cdots + b_m x^m \in \mathbb{Q}[x]$, write $b_0 = p/q$ with $p, q \in \mathbb{Z}$, and let ℓ be a prime not dividing q. I claim that $\frac{1}{\ell} \notin \operatorname{im}\psi$. Indeed, for any

$$f = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x]$$

we have

$$\varphi(f) = a_0 + a_1 g + a_2 g^2 + \cdots + a_n g^n,$$

whose constant term is

$$\begin{aligned} &a_0 + a_1 b_0 + a_1 b_0^2 + \cdots + a_n b_0^n \\ &\qquad = (a_0 q^n + a_1 p q^{n-1} + a_2 p^2 q^{n-2} + \cdots + a_n p^n)/q^n. \end{aligned}$$

Since ℓ does not divide q^n, this cannot equal $1/\ell$.

Alternatively, consider the map $r\colon \mathbb{Q}[x] \to \mathbb{Q}$ given by $r(g) = g(0)$, which is a surjective homomorphism. If $\varphi\colon \mathbb{Z}[y] \to \mathbb{Q}[x]$ were a surjective homomorphism then $r \circ \varphi\colon \mathbb{Z}[y] \to \mathbb{Q}$ would be a surjective homomorphism. So it is enough to argue that there is no surjective homomorphism $\mathbb{Z}[x] \to \mathbb{Q}$, which is similar to the argument above but maybe a little cleaner.

4. (a) Write down two or three elements of $\mathbb{Q}[x][y]$.
Solution: I'll chose

$$(x^2+4)y^2 + (2x+3)y + (x^2+1)$$
$$xy^3 + (x^2-1)y + 2x$$

(b) For any ring R, give an isomorphism $\varphi\colon R[x][y] \to R[y][x]$.
Solution: The point is that I want $\varphi(x^p y^q) = y^q x^p$. Given

$$f = g_0 + g_1 y + g_2 y^2 + \cdots + g_n y^n \in R[x][y],$$

let $m = \max\{\deg g_0, \deg g_1, \ldots, \deg g_n\}$, and for each i write

$$g_i = a_{i0} + a_{i1}x + a_{i2}x^2 + \cdots + a_{im}x^m \in R[x].$$

I define

$$\varphi(f) = h_0 + h_1 x + h_2 x^2 + \cdots + h_m x^m \in R[y][x],$$

where

$$h_j = a_{0j} + a_{1j}y + a_{2j}y^2 + \cdots + a_{nj}y^n \in R[y].$$

(c) Write down φ of your elements from part (a).
Solution:

$$(y^2+1)x^2 + (2y)x + (4y^2+3y+1)$$
$$-y + (y^3+2)x + yx^2$$

5. Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be a ring homomorphism.

   (a) Let $a \in \mathbb{Z} \subset \mathbb{R}$. Show that $\varphi(a) = a$.
   Solution: We have seen that $\varphi(0) = 0$ and $\varphi(1) = 1$. Then

   $$\varphi(2) = \varphi(1+1) = \varphi(1) + \varphi(1) = 1 + 1 = 2,$$

   and similarly $\varphi(3) = 3$, $\varphi(4) = 4$, and $\varphi(a) = a$ for all $a > 0$. (By induction if you like.) If $a < 0$ then

   $$\varphi(a) = \varphi(-(-a)) = -\varphi(-a) = -(-a) = a.$$

   (b) Let $a \in \mathbb{Q} \subset \mathbb{R}$. Show that $\varphi(a) = a$.
   Hint: Write $a = p/q$, and consider $q \cdot \varphi(a)$.
   Solution: Following the hint, we write $a = p/q$. Then

   $$q \cdot \varphi(a) = \varphi(q)\varphi(a) = \varphi(qa) = \varphi(p) = p,$$

   so $\varphi(a) = p/q$ as desired.

   (c) Let $a \in \mathbb{R}$. Show that if $a \geq 0$ then $\varphi(a) \geq 0$.
   Hint: Consider $\varphi(\sqrt{a})$.
   Solution: Following the hint, we observe that if $a \geq 0$ then there is a $b \in \mathbb{R}$ with $b^2 = a$. Then

   $$\varphi(a) = \varphi(b^2) = \varphi(b)^2 \geq 0.$$

   (d) Let $a, b \in \mathbb{R}$. Show that if $a \leq b$ then $\varphi(a) \leq \varphi(b)$.
   Solution: If $a \leq b$ then $b - a \geq 0$, so $\varphi(b-a) \geq 0$ by the previous part, so $\varphi(b) - \varphi(a) \geq 0$, so $\varphi(a) \leq \varphi(b)$.

   (e) Now let $a \in \mathbb{R}$ be arbitrary. Show that $\varphi(a) = a$.
   Hint: Consider $\{b \in \mathbb{Q} : b \leq a\}$ and/or $\{b \in \mathbb{Q} : b \geq a\}$.
   You can use facts from analysis without proof.
   Solution: We ignore the hint, and instead take the suggestion that I e-mailed to the group.
   Suppose on the contrary that there is an $a \in \mathbb{R}$ with $\varphi(a) \neq a$. If $\varphi(a) < a$, then by the density of the rationals we can choose $b \in \mathbb{Q}$ with $\varphi(a) < b < a$. But then $\varphi(b) \leq \varphi(a)$, so

   $$\varphi(a) < b = \varphi(b) \leq \varphi(a),$$

   so $\varphi(a) < \varphi(a)$ which is impossible. If $a < \varphi(a)$ then we get a similar contradiction. Thus $\varphi(a) = a$ for all $a \in \mathbb{R}$.

(f) In contrast, show that the map $\varphi\colon \mathbb{C} \to \mathbb{C}$ given by $\varphi(a+bi) = a - bi$ is a homomorphism.

Solution: First, $\varphi(1) = 1$.

Next,

$$\begin{aligned}
\varphi((a+bi)+(a'+b'i)) &= \varphi((a+a')+(b+b')i) \\
&= (a+a')-(b+b')i \\
&= (a-bi)+(a'-b'i) \\
&= \varphi(a+bi)+\varphi(a'+b'i).
\end{aligned}$$

Finally,

$$\begin{aligned}
\varphi((a+bi)(a'+b'i)) &= \varphi((aa'-bb')+(ab'+ba')i) \\
&= (aa'-bb')-(ab'+ba')i \\
&= (a-bi)(a'-b'i) \\
&= \varphi(a+bi)\varphi(a'+b'i).
\end{aligned}$$