Homework 2

Due Wednesday, October 12, 2016

1. Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ be the set of natural numbers, and define operations $\oplus$ and $\odot$ by

$$x \oplus y = \max(x, y) \qquad\qquad x \odot y = x + y.$$

   Which of the ring axioms hold for $\oplus$ and $\odot$, and which ones fail?

2. Let R and S be rings, and let $R \times S = \{(x, y) : x \in R, y \in S\}$ be their Cartesian product. Define a ring structure on $R \times S$ by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + y_1, x_2 + y_2)$$
$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 \cdot y_1, x_2 \cdot y_2)$$

   (a) What is 0 in $R \times S$? What is 1? What is $-(x, y)$?

   (b) List the elements of $\mathbb{Z}/2 \times \mathbb{Z}/3$. Which are zero-divisors? Which are units?

   (c) Show that if neither R nor S is the zero ring, then $R \times S$ has zero-divisors.

   (d) Show that $(x, y) \in R \times S$ is a zero-divisor if and only if x is zero or a zero-divisor in R, or y is zero or a zero-divisor in S.

   (e) Show that $(x, y) \in R \times S$ is a unit if and only if x is a unit in R and y is a unit in S.

3. Let R be a ring. Given $x \in R$, define a map $\lambda_x \colon R \to R$ by $\lambda_x(y) = xy$, and a map $\rho_x \colon R \to R$ by $\rho_x(y) = yx$.

   (a) Suppose that $x \neq 0$. Show that x is not a zero-divisor if and only if λ_x and ρ_x are injective.

   (b) Show that x is a unit if and only if λ_x and ρ_x are surjective.

   (c) Let R be a finite commutative ring. Show that either R is a field, or R has zero-divisors.

4. Let R be a ring, and take three generic elements of the ring of formal power series $R[[x]]$:

$$f = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$g = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots$$
$$h = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots ,$$

where $a_i, b_i, c_i \in R$.

(a) What is the coefficient of x^n in $f+g$? In $f \cdot g$?

(b) Show that multiplication is associative: $(f \cdot g) \cdot h = f \cdot (g \cdot h)$.

(c) Show that f is a unit in $R[[x]]$ if and only if a_0 is a unit in R.

5. In class we had a long discussion about units and zero-divisors in $M_2(\mathbb{Z})$; we argued that a matrix $A \in M_2(\mathbb{Z})$ is a unit if and only if $\det A = \pm 1$, and is a zero-divisor if and only if $\det A = 0$. Write a clean, concise proof of this.

6. What is one question you have about last week's lectures?