Solutions to Midterm 2

Let $X \subset \mathbb{R}^2$ be a connected, 1-dimensional manifold. You will prove that for almost every point $q \in \mathbb{R}^2$, the map $X \to \mathbb{R}$ given by $p \mapsto |p - q|^2$ is a Morse function. The same is true in higher dimensions, but the proof is harder.

You are given a surjective local diffeomorphism $\gamma \colon \mathbb{R} \to X$, which you can write in components as

$$\gamma(t) = (x(t), y(t)).$$

1. (5 points) Fix a point $(a, b) \in \mathbb{R}^2$, and let $f \colon \mathbb{R} \to \mathbb{R}$ be defined by

   $$f(t) = (x(t) - a)^2 + (y(t) - b)^2,$$

   which is the square of the distance from $\gamma(t)$ to (a, b).

   Compute $f'(t)$.

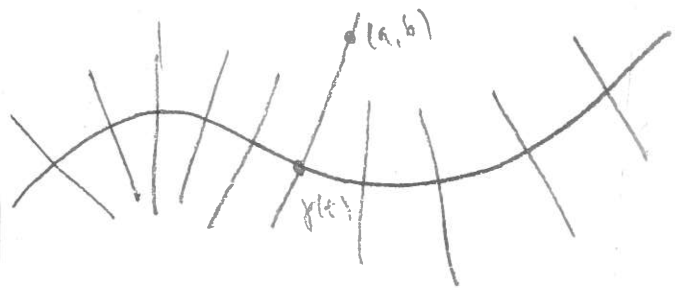
   Solution:

   $$f'(t) = 2(x(t) - a)x'(t) + 2(y(t) - b)y'(t).$$

2. (10 points) Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$F(t,u) = (x(t) + uy'(t), y(t) - ux'(t)).$$

Show that $f'(t) = 0$ if and only if there is a $u \in \mathbb{R}$ with $F(t,u) = (a,b)$. (Geometric remark: If t is fixed and u varies then $F(t,u)$ traces out the line that's perpendicular to the curve at $\gamma(t)$. So you are proving that $f'(t) = 0$ if and only if this line passes through (a,b).)



Solution: First suppose that $F(t,u) = (a,b)$. Then $a = x(t) + uy'(t)$ and $b = y(t) - ux'(t)$, so

$$\begin{aligned} f'(t) &= 2(x(t)-a)x'(t) + 2(y(t)-b)y'(t) \\ &= 2(-uy'(t))x'(t) + 2(ux'(t))y'(t) \\ &= 0. \end{aligned}$$

There are several ways to do the converse, but here is one. Observe that

$$f'(t) = 2\det\begin{pmatrix} x(t)-a & y(t)-b \\ -y'(t) & x'(t) \end{pmatrix},$$

so if $f'(t) = 0$ then this 2×2 matrix is singular. Because γ is a local diffeomorphism, $\gamma'(t) \neq 0$, so the second row is not zero, so the first row must be a multiple of the second row. Thus there is a $u \in \mathbb{R}$ such that $x(t) - a = -uy'(t)$ and $y(t) - b = ux'(t)$, so $(a,b) = F(t,u)$ as desired.

3. (5 points) Compute $f''(t)$.

   Solution:

$$f''(t) = 2x'(t)^2 + 2(x(t) - a)x''(t) + 2y'(t)^2 + 2(y(t) - b)y''(t).$$

4. (5 points) Compute $dF_{(t,u)}$.

   Solution:
$$dF_{(t,u)} = \begin{pmatrix} x'(t) + uy''(t) & y'(t) \\ y'(t) - ux''(t) & -x'(t) \end{pmatrix}$$

5. (10 points) Suppose that $f'(t) = 0$, so by problem 2 there is a $u \in \mathbb{R}$ with $F(t, u) = (a, b)$.

   Show that (t, u) is a critical point of F if and only if $f''(t) = 0$.

   Solution: In problem 2 we saw that $x(t) - a = -uy'(t)$ and $y(t) - b = ux'(t)$. Plugging this into our expression for $f''(t)$, we get

$$f''(t) = 2x'(t)^2 + 2(-uy'(t))x''(t) + 2y'(t)^2 + 2(ux'(t))y''(t),$$

   which is exactly $-2 \det dF_{(t,u)}$. Thus $f''(t) = 0$ if and only if $dF_{(t,u)}$ is singular.

6. (5 points) Define what it means for f to be a Morse function.

   Hint: Because f is a function of one variable, you can simplify the the definition a lot.

   Solution: For every $t \in \mathbb{R}$ with $f'(t) = 0$ we have $f''(t) \neq 0$.

7. (5 points) State Sard's theorem for the map F.

   Solution: The set of critical values of F has measure zero in $\mathbb{R}^2$.

8. (5 points) Now let (a, b) vary, and consider the set of points $(a, b) \in \mathbb{R}^2$ for which the function f defined in problem 1 is *not* a Morse function. Show that this set has measure zero.

   Hint: Don't work hard, just use what you've already done.

   Solution: From problems 2 and 5 we see that f is not a Morse function if and only if (a, b) is a critical value of F. By Sard's theorem, the set of all such (a, b) has measure zero in $\mathbb{R}^2$.