Midterm 1

Friday, Feburary 7, 2020

Each part is worth 5 points, for a total of 40 points.

1. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ and $g\colon \mathbb{R}^3 \to \mathbb{R}$ be given by

$$f(x,y,z) = x^2 + y^2 + z^2 \qquad\qquad g(x,y,z) = x^2 + y^2 - z^2.$$

On the last homework you showed that the sphere $f^{-1}(a)$ and the hyperboloid $g^{-1}(1)$ are transverse unless $a = 1$. Now consider the function

$$F\colon \mathbb{R}^3 \to \mathbb{R}^2$$

given by

$$F(x,y,z) = (x^2 + y^2 + z^2,\; x^2 + y^2 - z^2),$$

that is, $F = f \times g$.

(a) Find the derivative of F.
Hint: This is a 2×3 matrix whose entries depend on x, y, and z.

(b) Define what it means for a point $(x,y,z) \in \mathbb{R}^3$ to be a *critical point* of F.

(c) Describe the set of critical points of F using equations, and draw a picture.
Hint: It's the union of a plane and a line.

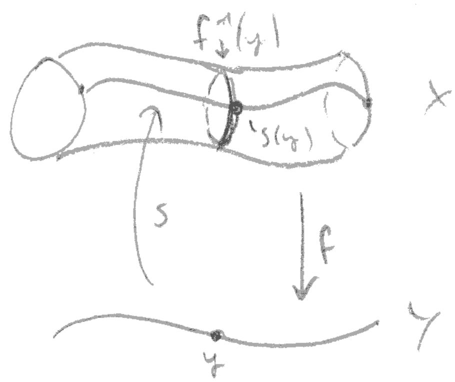
(d) Define what it means for a point $(a,b) \in \mathbb{R}^2$ to be a *critical value* of F.

(e) Describe the set of critical values of F using equations, and draw a picture.
Hint: It's the union of two half-lines.
Sanity check: From the homework you know (or can believe) that $(a,1)$ is a critical value if and only if $a = 1$.

(f) Over the critical value $(1,1)$, the sphere $f^{-1}(1)$ and hyperboloid $g^{-1}(1)$ are tangent along a circle, as you saw in on the homework. What happens over the critical value $(1,-1)$? Describe it in words and draw a picture.

(exam continues on the next page)

2. Let $f\colon X \to Y$ be a smooth map between manifolds. A *section* or *cross-section* of f is a map $s\colon Y \to X$ such that $f \circ s = 1_Y$, that is, $f(s(y)) = y$ for all $y \in Y$, or equivalently, $s(y)$ is in the fiber $f^{-1}(y)$ for all $y \in Y$.



(a) Show that if s is a smooth section of f, then for any $y \in Y$ the point $s(y) \in X$ is a regular point of f; or to use another term, f is a submersion at $s(y)$.

Hint: Apply the chain rule to $f \circ s$.

(b) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = x^2 - y^2$.

Sketch the fibers $f^{-1}(t)$ for $t = -2, -1, 0, 1, 2$.

The origin $(0, 0) \in \mathbb{R}^2$ is a critical point of f, so by part (a) there is no smooth section $s\colon \mathbb{R} \to \mathbb{R}^2$ of f with $s(0) = (0, 0)$. But exhibit a *continuous* section with $s(0) = (0, 0)$.