Solutions to Homework 6

7. *Show that the* antipodal map $x \mapsto -x$ *of* $S^k \to S^k$ *is homotopic to the identity if* k *is odd.*

   Write $k = 2n$, and define a map $F\colon S^{2n-1} \times I \to S^{2n-1}$ by

$$F(x_1, y_1, x_2, y_2, \dots, x_n, y_n, t) =$$

$$\begin{pmatrix} \cos \pi t & -\sin \pi t & & & & & \\ \sin \pi t & \cos \pi t & & & & & \\ & & \cos \pi t & -\sin \pi t & & & \\ & & \sin \pi t & \cos \pi t & & & \\ & & & & \ddots & & \\ & & & & & \cos \pi t & -\sin \pi t \\ & & & & & \sin \pi t & \cos \pi t \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ x_n \\ y_n. \end{pmatrix}$$

   This is clearly smooth, and we check that it takes values in S^{2n-1}. At $t = 0$ it is the identity map, and at $t = 1$ it is the antipodal map.

   We can package this much more nicely using complex numbers: we have

$$S^{2n-1} = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 = 1\},$$

   and

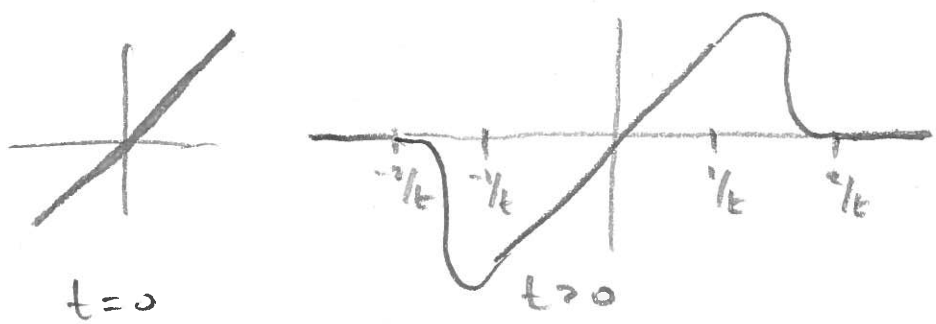
$$F(z_1, \dots, z_n, t) = (e^{i\pi t} z_1, \dots, e^{i\pi t} z_n).$$

9. *Prove that the Stability Theorem is false on noncompact domains. Here's one counterexample... Let $\rho\colon \mathbb{R} \to \mathbb{R}$ be a function with $\rho(s) = 1$ if $|s| < 1$ and $\rho(s) = 0$ if $|s| > 2$. Define $f_t\colon \mathbb{R} \to \mathbb{R}$ by $f_t(x) = x\rho(tx)$. Verify that this is a counterexample to all six parts of the theorem.*

We note that for any map $f\colon \mathbb{R} \to \mathbb{R}$, the following are equivalent:

- $f'(x) \neq 0$ for all $x \in \mathbb{R}$,
- f is a local diffeomorphism,
- f is an immersion,
- f is a submersion.

The graph of f_t looks like this:



The point is that f_t is badly behaved for all $t > 0$, but the bad parts run off to zero "suddenly" at $t = 0$.

If $t = 0$ then f_t is the identity map, so it is a local diffeomorphism, an immersion, a submersion, transverse to the submanifold $Z = \{0\} \subset \mathbb{R}$, an embedding, and a diffeomorphism. If the Stability Theorem held for non-compact domains, then there would be an $\epsilon > 0$ such that f_t had the same properties for all $t < \epsilon$.

But if $t > 0$ then there is a point, say $x = 3/t$, near which f_t is constantly zero, so $f'(3/t) = 0$, so f_t is not a local diffeomorphism, an immersion, or a submersion. Moreover f_t is not injective, so it is not an embedding or a diffeomorphism. Finally f_t is not transverse to the submanifold $Z = \{0\} \subset \mathbb{R}$, because $3/t \in f_t^{-1}(0)$ but the image of the derivative of f_t at $3/t$ is the zero vector space in $T_0(\mathbb{R}) = \mathbb{R}$, and the tangent space $T_0(Z) \subset T_0(\mathbb{R})$ is also the zero vector space, and these do not span $T_0(\mathbb{R})$.

4. *Prove that the rational numbers have measure zero in* $\mathbb{R}$*, even though they are dense.*

   You could say that the rational numbers are a countable union of sets of measure zero (points), so they have measure zero by the argument on page 40. Or you could just spell it out. Enumerate the rational numbers in a sequence $r_0, r_1, r_2, \ldots$. Let $\epsilon > 0$ be given. For each $i = 0, 1, 2, \ldots$, consider the interval $(r_i - \epsilon/2^{i+1}, r_i + \epsilon/2^{i+1})$, whose length is $\epsilon/2^i$. Then $\mathbb{Q} = \{r_0, r_1, \ldots\}$ is contained in the union of these intervals, and the sum of their lengths is ϵ.

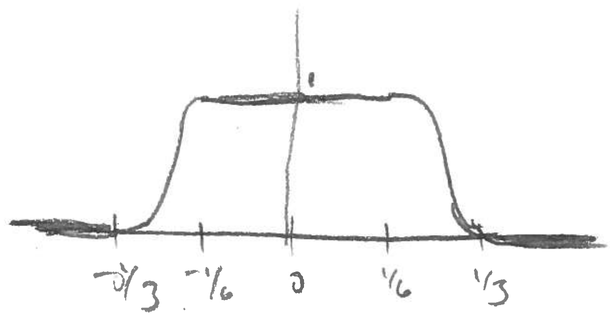
   You don't have to prove that $\mathbb{Q}$ is dense in $\mathbb{R}$, but if you want to, it goes like this. Let $U \subset \mathbb{R}$ be open, so U contains some open interval (a, b). Let n be an integer greater than $1/(b-a)$, so $1/n < b - a$. Let m be the smallest integer greater than an. Then $a < m/n$, and $m \leq an + 1$, so

   $$m/n \leq a + 1/n < a + (b-a) < b,$$

   so $m/n \in U$. Thus $U \cap \mathbb{Q} \neq \varnothing$.

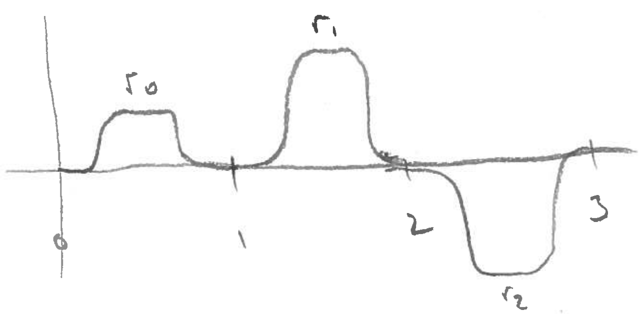
5. *Exhibit a smooth map* $f\colon \mathbb{R} \to \mathbb{R}$ *whose set of critical values is dense.*

By §1 #18(c), there is a smooth function $F\colon \mathbb{R} \to \mathbb{R}$ such that $F(x) = 1$ for $|x| \le \frac{1}{6}$ and $F(x) = 0$ for $|x| \ge \frac{1}{3}$:



Again enumerate the rational numbers in a sequence $r_0, r_1, \dots,$ and let

$$G(x) = \sum_{i=0}^{\infty} r_i \cdot F(x - (i + \tfrac{1}{2})).$$



The sum is legitimate because at any given x, only one term in the sum is non-zero. The function is smooth because on any interval $(i, i+1)$ it is a translate of the smooth function F, and on an interval $(i - \frac{1}{6}, i + \frac{1}{6})$ it is constantly zero. On an interval $(i + \frac{1}{3}, i + \frac{2}{3})$ it is constantly equal to r_i, so the derivative vanishes on that interval and the value r_i is a critical value.