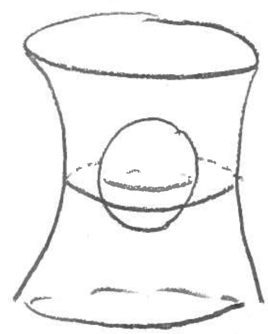
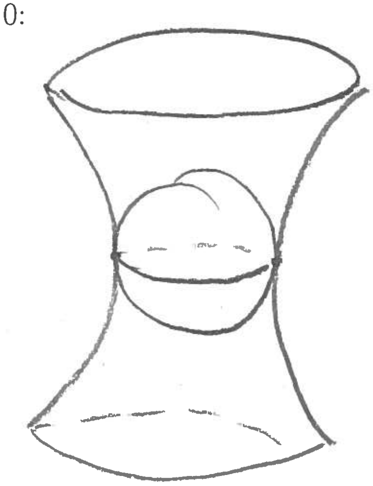
Solutions to Homework 5

8. *For which values of a does the hyperboloid $x^2 + y^2 - z^2 = 1$ intersect the sphere $x^2 + y^2 + z^2 = a$ transversely? What does the intersection look like for different values of a?*

   The intersection is empty, and in particular, transverse, if $a < 1$:

   If $a = 1$ then the two surfaces are tangent along the circle cut out by $x^2 + y^2 = 1$ and $z = 0$:

This is not transverse at any point: the tangent space to the hyperbola at $(x, y, 0)$ is the kernel of

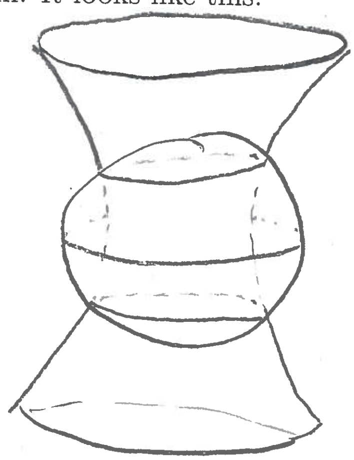
$$\begin{pmatrix} 2x & 2y & -2z \end{pmatrix} = \begin{pmatrix} 2x & 2y & 0 \end{pmatrix},$$

and the tangent space to the sphere is the kernel of

$$\begin{pmatrix} 2x & 2y & 2z \end{pmatrix} = \begin{pmatrix} 2x & 2y & 0 \end{pmatrix},$$

which is the same.

If $a > 1$, the intersection consists of the two circles cut out by $x^2 + y^2 = (a+1)/2$ and $z = \pm\sqrt{(a-1)/2}$; to see this, add the two equations, then subtract them. It looks like this:



The intersection is transverse at every point: the tangent space to the hyperbola at $(x, y, \sqrt{(a-1)/2})$ is the kernel of

$$\begin{pmatrix} 2x & 2y & -2z \end{pmatrix} = \begin{pmatrix} 2x & 2y & \mp 2\sqrt{(a-1)/2} \end{pmatrix},$$

and the tangent space to the sphere is the kernel of

$$\begin{pmatrix} 2x & 2y & 2z \end{pmatrix} = \begin{pmatrix} 2x & 2y & \pm 2\sqrt{(a-1)/2} \end{pmatrix}.$$

These two kernels span $\mathbb{R}^3$ if and only if they are not the same, which is true if and only if the two row vectors are linearly independent. Since the third entries are different, the row vectors are linearly dependent if and only if $x = y = 0$, which would contradict $x^2 + y^2 = (a+1)/2$.

1. *Suppose that $f_0, f_1 \colon X \to Y$ are homotopic. Show that there exists a homotopy $\tilde{F} \colon X \times I \to Y$ such that $\tilde{F}(x,t) = f_0(x)$ for all $t \in [0, 1/4]$ and $\tilde{F}(x,t) = f_1(x)$ for all $t \in [3/4, 1]$.*

   Following the hint, we let $\rho \colon \mathbb{R} \to \mathbb{R}$ be the function h constructed in §1.1 #18(b) with $a = 1/4$ and $b = 3/4$. Thus for $t \leq 1/4$ we have $\rho(t) = 0$, for $1/4 < t < 3/4$ we have $0 < \rho(t) < 1$, and for $t \geq 3/4$ we have $\rho(t) = 1$.

   Now if $F \colon X \times I \to Y$ is any homotopy between f_0 and f_1, then the map $\tilde{F} \colon X \times I \to Y$ given by $\tilde{F}(x,t) = F(x, \rho(t))$ has the required properties.

2. *Prove that [smooth] homotopy is an equivalence relation: if $f \sim g$ and $g \sim h$ then $f \sim h$.*

   You can prove that homotopy is reflexive and symmeric if you want, although we did it in lecture. Reflexive: For any $f \colon X \to Y$, the map $F(x,t) = f(x)$ is a homotopy from f to f. Symmetric: If $F(x,t)$ is a homotopy from f to g then $F(x, 1-t)$ is a homotopy from g to f.

   But as the problem suggests, transitivity is the interesting thing. Let F be a homotopy from f to g, let G be a homotopy from g to h, and let $\tilde{F}$ and $\tilde{G}$ be constructed from F and G as in the previous problem. I claim that the map $H \colon X \times I \to Y$ given by

$$H(x,t) = \begin{cases} \tilde{F}(x, 2t) & \text{if } t \leq 1/2 \\ \tilde{G}(x, 2t-1) & \text{if } t \geq 1/2. \end{cases}$$

   is a homotopy from f to h. The subtlety is in showing that H is smooth; observe that if we had defined H using F and G in place of $\tilde{F}$ and $\tilde{G}$ then the result would be continuous but probably not smooth.

   The map H is smooth on $X \times [0, 1/2)$ because it is a composition of smooth functions $\tilde{F}$ and $(x,t) \mapsto (x, 2t)$. Similarly, it is smooth on $X \times (1/2, 1]$. It is smooth on $X \times (3/8, 5/8)$ because $H(x,t) = g(x)$ for $3/8 < t < 5/8$. Thus it is smooth at every point of $X \times I$.