Solutions to Homework 1

7. *Prove that the coordinate axes in $\mathbb{R}^2$ are not a manifold.*

   Let $X = \{(x,y) \in \mathbb{R}^2 : xy = 0\}$ be the coordinate axes. There is no neighborhood of 0 that is homeomorphic to an open set in $\mathbb{R}^k$ for any k, much less diffeomorphic. To see this, let $U \subset X$ be an open set containing 0, and let U_0 be the connected component of U that contains 0. Then $U_0 \setminus 0$ has four connected components. But if $V_0 \subset \mathbb{R}^n$ is a connected open set and $p \in V_0$, then $V_0 \setminus p$ is either empty (if $n = 0$), or has two connected components (if $n = 1$), or is connected (if $n \geq 2$).

10. *"The"* torus *is the set of points in $\mathbb{R}^3$ at a distance b from the circle of radius a in the xy-plane, where $0 < b < a$. Prove that these tori are all diffeomorphic to $S^1 \times S^1$.*

    Let

    $$X = S^1 \times S^1 = \{(x,y,z,w) \in \mathbb{R}^4 : x^2 + y^2 = 1, z^2 + w^2 = 1\}.$$

    Let $Y \subset \mathbb{R}^3$ be the torus described in the problem, for some $0 < b < a$. Define a map $f \colon X \to Y$ by

    $$f(x,y,z,w) = (ax + bxz, ay + byz, bw).$$

    Clearly f is smooth, because its components are polynomials. To see that it takes values in Y, fix $(x,y) \in S^1$; then as $(z,w) \in S^1$ varies, we see that $f(x,y,z,w)$ lies in the plane spanned by $(x,y,0)$ and $(0,0,1)$, and traces out the circle of radius b around the point $(ax, ay, 0)$ in that plane.

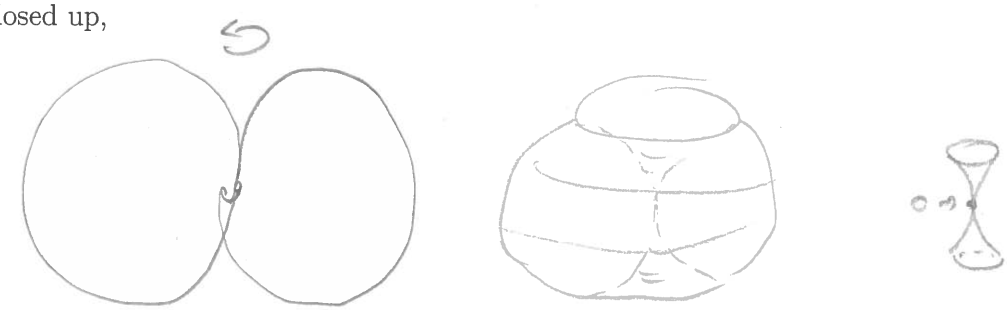
To see that f is a diffeomorphism, we should write down its inverse and see that it is smooth. Define $g\colon Y \to X$ by

$$g(x, y, z) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, \frac{\sqrt{x^2+y^2}-a}{b}, \frac{z}{b} \right).$$

This is smooth because the constraint $b < a$ forces $x^2 + y^2 > 0$ for all $(x, y, z) \in Y$. I omit the straightforward but tedious verification that $g(Y) \subset X$, that $g \circ f = 1_X$, and that $f \circ g = 1_Y$.

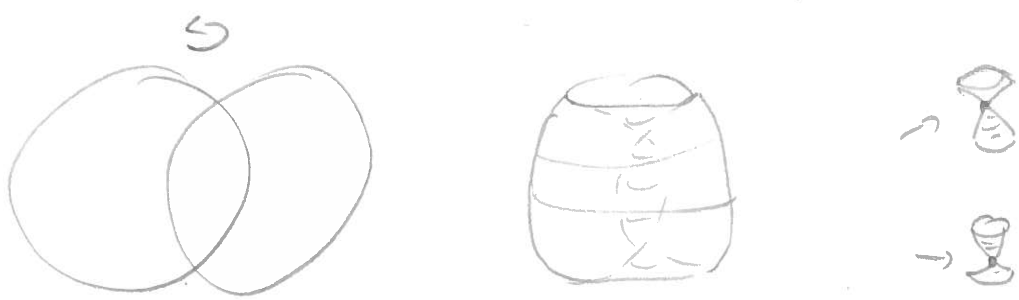
Also draw the cases $b = a$ and $b > a$; why are these not manifolds?

The case $b = a$ looks like a bagel that has risen so much that the hole closed up,



and it fails to be a manifold at the origin: for every open set $U \ni 0$, there is a connected open set U' with $0 \in U' \subset U$ such that $U' \setminus 0$ is disconnected, and $U' \setminus p$ is conncted for every other $p \in U'$. But this is not true of an open set in $\mathbb{R}^k$.

The case $b > a$ looks like this,



and fails to be a manifold at the points $(0, 0, \pm\sqrt{b^2 - a^2})$, for the same reason.

11. *Show that one cannot parametrize the k-sphere S^k by a single parametrization.*

Let $U \subset \mathbb{R}^k$ be open, let V be an open subset of the sphere $S^k \subset \mathbb{R}^{k+1}$, and let $\phi\colon U \to V$ be a diffeomorphism (a parametrization). If V is all of S^k then V is compact, and ϕ is in particular a homeomorphism, so U is compact, hence closed and bounded in $\mathbb{R}^k$. But U is also open, and non-empty, which is impossible because $\mathbb{R}^k$ is connected.

18. (a) *An extermely useful function $f\colon \mathbb{R} \to \mathbb{R}$ is*

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Prove that f is smooth. [You may use $e^{-1/x}$ if you prefer.]

Away from 0, f is a composition of smooth functions, hence is smooth, so we need only show that f is smooth at 0.

First let us prove that $f'(0) = 0$. The limit from the left is clear, and from the right we have

$$\begin{aligned} \lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^+} \frac{\exp(-h^{-2})}{h} &= \lim_{h\to 0^+} \frac{h^{-1}}{\exp(h^{-2})} \\ &= \lim_{t\to\infty} \frac{t}{\exp(t^2)}, \end{aligned}$$

where in the last step we substituted $t = 1/h$. By l'Hôpital's rule, this is equal to

$$\lim_{t\to\infty} \frac{1}{2t\exp(t^2)} = 0.$$

So we have

$$f'(x) = \begin{cases} 2x^{-3}\exp(-x^{-2}) & x > 0 \\ 0 & x \leq 0. \end{cases}$$

We see that $\lim_{x\to 0^+} f'(x) = 0$ by a similar substitution $t = 1/x$ and two applications of l'Hôpital's rule, so f' is continuous at 0.

Now we see the pattern. For $x > 0$, we have

$$\begin{aligned} f'(x) &= 2x^{-3}\exp(-x^{-2}) \\ f''(x) &= (4x^{-6} - 6x^{-4})\exp(-x^{-2}) \\ f'''(x) &= (8x^{-9} - 36x^{-7} + 24x^{-5})\exp(-x^{-2}), \end{aligned}$$

and in general we see that the n^{th} derivative $f^{(n)}(x)$ can be written as $p_n(x^{-1})\exp(-x^{-2})$ for some polynomial p_n.* Each of these goes to 0 as $x \to 0^+$, using the same substitution $t = 1/x$ and several applications of l'Hôpital's rule. So to prove that f is smooth, it is enough to prove that $f^{(n)}(0) = 0$ for every n, which we will do by induction.

We have seen that $f(0) = 0$ and $f'(0) = 0$. If $f^{(n)}(0) = 0$, then

$$\begin{aligned} \lim_{h\to 0^+} \frac{f^{(n)}(h) - f^{(n)}(0)}{h} &= \lim_{h\to 0^+} \frac{p_n(h^{-1})\exp(-h^{-2})}{h} \\ &= \lim_{h\to 0^+} \frac{h^{-1}\cdot p_n(h^{-1})}{\exp(h^{-2})} = \lim_{t\to\infty} \frac{t\cdot p_n(t)}{\exp(t^2)} = 0, \end{aligned}$$

so $f^{(n+1)}(0) = 0$ as desired.

(b) *Show that $g(x) = f(x-a)f(b-x)$ is a smooth function, positive on (a,b) and zero elsewhere. (Here $a < b$.) Then*

$$h(x) = \frac{\int_{-\infty}^{x} g(t)\,dt}{\int_{-\infty}^{\infty} g(t)\,dt}$$

is a smooth function satisfying $h(x) = 0$ for $x \le a$, $h(x) = 1$ for $x \ge b$, and $0 < h(x) < 1$ for $x \in (a,b)$.

We see that $f(x-a)$ and $f(b-x)$ are compositions of smooth functions, hence are smooth, so their product is smooth. If $a < x < b$ then $x - a > 0$, so $f(x-a) > 0$, and $b - x > 0$, so $f(b-x) > 0$, so their product $g(x) > 0$. If $x \le a$ then $x - a \le 0$, so $f(x-a) = 0$, so $g(x) = 0$. Similarly if $x \ge b$ then $g(x) = 0$.

*If you want to prove this carefully by induction, $p_0(x) = 1$ is a polynomial, and we find that $p_{i+1}(x) = 2x^3p_i(x) - x^2p_i'(x)$, so if p_i is a polynomial then p_{i+1} is as well.

The fundamental theorem of calculus gives

$$h'(x) = \frac{g(x)}{\int_{-\infty}^{\infty} g(t)\,dt}$$

which is a scalar multiple of a smooth function $g(x)$. Thus $h^{(n)}(x)$ is the same scalar multiple of $g^{(n-1)}(x)$, which is continuous.

Because $g(t) = 0$ for $t \le a$, the integral in the numerator is zero for $x \le a$, so $h(x)$ is zero there as well.

Because $g(t) = 0$ for $t \ge b$, the integrals in the numerator and denominator are equal for $x \ge b$, so $h(x) = 1$ there.

Because g is continuous and $g(t) > 0$ for $a < t < b$, we see that $\int_a^x g(t)\,dt > 0$ for $a < x < b$, so $h(x) - h(a) = 0$, so $h(x) > 0$; and $\int_x^b g(t)\,dt > 0$, so $h(b) - h(x) > 0$, so $h(x) < 1$.

(c) *Now construct a smooth function on $\mathbb{R}^k$ that equals 1 on the ball of radius a, zero outside the ball of radius b, and is strictly between 0 and 1 at intermediate points. (Here $0 < a < b$.)*

Define

$$F(x_1, \ldots, x_k) = 1 - h\left(\sqrt{x_1^2 + \cdots + x_k^2}\right).$$

To see that F is smooth, observe that away from the origin it is a composition of smooth functions, and near the origin it agrees with the constant function 1. The other desired properties are immediate from what we proved about h.