Solutions to Midterm 2

1. Let (X, d_X) and (Y, d_Y) be metric spaces.

   You know that a map $f\colon X \to Y$ is called *continuous* if for every $p \in X$ and every $\epsilon > 0$ there is a $\delta > 0$ such that for all $q \in X$ with $d_X(p, q) < \delta$, we have $d_Y(f(p), f(q)) < \epsilon$.

   On the other hand, f is called *uniformly continuous* if for every $\epsilon > 0$ there is a $\delta > 0$ such that for all $p, q \in X$ with $d_X(p, q) < \delta$, we have $d_Y(f(p), f(q)) < \epsilon$. The difference is the order of the quantifiers.

   (a) (3 points) Define what it means for a sequence $p_1, p_2, \ldots \in X$ to be *Cauchy*.

   Solution: For every $\epsilon > 0$ there is a natural number N such that for all $m, n \geq N$ we have $d_X(p_m, p_n) < \epsilon$.

   (b) (3 points) Prove that if $p_1, p_2, \ldots \in X$ is Cauchy and f is uniformly continuous, then $f(p_1), f(p_2), \ldots \in Y$ is Cauchy.

   Solution: Let $\epsilon > 0$ be given. Because f is uniformly continuous, there is a $\delta > 0$ such that for all $p, q \in X$ with $d_X(p, q) < \delta$ we have $d_Y(f(p), f(q)) < \epsilon$. Because the sequence p_n is Cauchy, there is an N such that for all $m, n \geq N$ we have $d_X(p_m, p_n) < \delta$, and thus $d_X(f(p_m), f(p_n)) < \epsilon$.

   (c) (3 points) Let $X = (0, \infty)$ with the usual metric inherited from $\mathbb{R}$, let $Y = \mathbb{R}$ with the usual metric, and let $f\colon X \to Y$ be given by $f(x) = \log x$. Give an example of a Cauchy sequence $p_1, p_2, \ldots \in X$ such that $f(p_1), f(p_2), \ldots \in Y$ is not Cauchy.

   Solution: The sequence $e^{-1}, e^{-2}, e^{-3}, \ldots \in \mathbb{R}$ converges to 0, so it is Cauchy in the usual metric, and it is still Cauchy when regarded as a sequence in $(0, \infty)$. But f maps it to $-1, -2, -3, \ldots$ which is not Cauchy.

2. (5 points) Prove that $\mathbb{R}^2$ cannot be written as a countable union of straight lines.

   Solution: A line is nowhere dense in the Euclidean metric: it is closed, but any open ball around any point spills out of the line, so the interior of the line interior is empty. We have seen that $\mathbb{R}^2$ is complete in the Euclidean metric. The Baire category theorem states that in a complete metric space, a countable union of nowhere dense sets has empty interior; in particular, it is not the whole space.

3. (a) (3 points) Let X be a topological space. Define what it means for a subset D to be *dense* in X.

      (If you only remember the definition in a metric space, the answer for topological spaces is the same.)

      Solution: Here are three possibile answers: the closure of D is X; the only closed set containing D is X; every non-empty open set $U \subset X$ meets D.

   (b) (3 points) Prove that $\mathbb{Q} \subset \mathbb{R}$ is dense in the lower limit topology, where open sets are unions of intervals of the form $[a, b)$.

      (Feel free to assert any major facts about $\mathbb{Q}$ without proof.)

      Solution: Let $U \subset \mathbb{R}$ be a non-empty open subset in the lower limit topology. Then U contains an interval of the form $[a, b)$ with $a < b$. We know that there is a rational number r with $a < r < b$. Thus $U \cap \mathbb{Q} \neq \varnothing$.

   (c) (5 points) Let $f\colon X \to Y$ be continuous and surjective. Prove that if D is dense in X, then $f(D)$ is dense in Y.

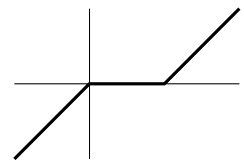
      Solution: Let $V \subset Y$ be a non-empty open subset. Because f is surjective, $f^{-1}(V)$ is not empty, and because f is continuous, $f^{-1}(V)$ is open in X. Because D is dense in X, $f^{-1}(V) \cap D \neq \varnothing$. Thus $V \cap f(D) \neq \varnothing$, so D is dense in Y.

      Or if you prefer, you could let $G \subset Y$ be a closed subset containing $f(D)$. Then $f^{-1}(G)$ contains D. Because f is continuous, $f^{-1}(G)$ is closed. Because D is dense in X, we have $f^{-1}(G) = X$. Because f is surjective, we conclude that $G = Y$.

(d) (3 points) Give an example of a continuous map $f\colon X \to Y$ and a dense subset $D \subset Y$ such that $f^{-1}(D)$ is not dense in X.

Solution: If you want a surjective map, you could let $X = Y = \mathbb{R}$ with the usual topology, and let f be the piecewise linear function

$$f(x) = \begin{cases} x & \text{if } x \le 0, \\ 0 & \text{if } 0 \le x \le 1, \\ x-1 & \text{if } x \ge 1, \end{cases}$$



and let $D = \mathbb{R} \setminus \mathbb{Q}$. Then $f^{-1}(D) = \mathbb{R} \setminus \mathbb{Q} \setminus [0,1]$, whose closure is $(-\infty, 0] \cup [1, \infty)$, not all of $\mathbb{R}$.

If you want an injective map, you could let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $f(x) = (x, 0)$ be the inclusion of the x-axis, and $D = (\mathbb{R} \setminus \mathbb{Q})^2$. Then $f^{-1}(D)$ is empty.

Of course there are many other possibilities.

4. (a) (3 points) Let X be a topological space and $Y \subset X$. Define what it means for a subset $V \subset Y$ to be open in the subspace topology.

Solution: There is an open set $U \subset X$ such that $V = U \cap Y$.

(b) (3 points) Put the usual topology on $\mathbb{R}$. Prove that the subspace topology on $\mathbb{Q}$ is not the discrete topology.

(Recall that the discrete topology is the one where every subset is open, or equivalently, every one-point set is open.)

Solution: Let us prove that the one-point set $\{0\}$ is not open in the subspace topology on $\mathbb{Q}$, that is, it is not of the form $U \cap \mathbb{Q}$ for some open set $U \subset \mathbb{R}$. If $U \subset \mathbb{R}$ is open and $0 \in U$, then there is an $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subset U$. Choose an integer $n > 1/\epsilon$; then $1/n \in (-\epsilon, \epsilon) \cap \mathbb{Q}$, so $1/n \in U \cap \mathbb{Q}$. Thus $U \cap \mathbb{Q}$ consists of more than just 0.

(c) (3 points) Give an example of an infinite subset $A \subset \mathbb{R}$ for which the subspace topology is the discrete topology.

Solution: Take $A = \mathbb{Z}$. For every $n \in \mathbb{Z}$, we have

$$\{n\} = (n - \tfrac{1}{2},\, n + \tfrac{1}{2}) \cap \mathbb{Z},$$

so every one-point subset is open in the subspace topology.