Midterm 2

Friday, November 21, 2025

Use your own notebook paper, or get some from me. I encourage you to write in pencil. There are 37 points in total.

1. Let (X, d_X) and (Y, d_Y) be metric spaces.

   You know that a map $f\colon X \to Y$ is called *continuous* if for every $p \in X$ and every $\epsilon > 0$ there is a $\delta > 0$ such that for all $q \in X$ with $d_X(p,q) < \delta$, we have $d_Y(f(p), f(q)) < \epsilon$.

   On the other hand, f is called *uniformly continuous* if for every $\epsilon > 0$ there is a $\delta > 0$ such that for all $p, q \in X$ with $d_X(p,q) < \delta$, we have $d_Y(f(p), f(q)) < \epsilon$. The difference is the order of the quantifiers.

   (a) (3 points) Define what it means for a sequence $p_1, p_2, \ldots \in X$ to be *Cauchy*.

   (b) (3 points) Prove that if $p_1, p_2, \ldots \in X$ is Cauchy and f is uniformly continuous, then $f(p_1), f(p_2), \ldots \in Y$ is Cauchy.

   (c) (3 points) Let $X = (0, \infty)$ with the usual metric inherited from $\mathbb{R}$, let $Y = \mathbb{R}$ with the usual metric, and let $f\colon X \to Y$ be given by $f(x) = \log x$. Give an example of a Cauchy sequence $p_1, p_2, \ldots \in X$ such that $f(p_1), f(p_2), \ldots \in Y$ is not Cauchy.

2. (5 points) Prove that $\mathbb{R}^2$ cannot be written as a countable union of straight lines.

(Continued on the other side.)

3. (a) (3 points) Let X be a topological space. Define what it means for a subset D to be *dense* in X.

   (If you only remember the definition in a metric space, the answer for topological spaces is the same.)

   (b) (3 points) Prove that $\mathbb{Q} \subset \mathbb{R}$ is dense in the lower limit topology, where open sets are unions of intervals of the form $[a, b)$.

   (Feel free to assert any major facts about $\mathbb{Q}$ without proof.)

   (c) (5 points) Let $f\colon X \to Y$ be continuous and surjective. Prove that if D is dense in X, then $f(D)$ is dense in Y.

   (d) (3 points) Give an example of a continuous map $f\colon X \to Y$ and a dense subset $D \subset Y$ such that $f^{-1}(D)$ is not dense in X.

4. (a) (3 points) Let X be a topological space and $Y \subset X$. Define what it means for a subset $V \subset Y$ to be open in the subspace topology.

   (b) (3 points) Put the usual topology on $\mathbb{R}$. Prove that the subspace topology on $\mathbb{Q}$ is not the discrete topology.

   (Recall that the discrete topology is the one where every subset is open, or equivalently, every one-point set is open.)

   (c) (3 points) Give an example of an infinite subset $A \subset \mathbb{R}$ for which the subspace topology is the discrete topology.